Intervention with Voluntary Participation in Global Games*

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Abstract

We analyze a canonical binary-action coordination game with incomplete information under the global games framework. The heterogeneity in agents' belief allows us to propose a novel stimulus program for a policy maker to reduce coordination failure. Compared with conventional government-guarantee type of programs, it incurs lower cost of implementation and is more robust to moral hazard problems. Specifically, our proposed stimulus program is a subsidy-tax offer to agents who take the efficient action. It has voluntary participation and screens for the marginal agents. In equilibrium, only a small mass of "pivotal agents" receiving medium signals self-select to participate in the program. However, the effect is amplified by higher-order beliefs, and coordination failures can be significantly reduced. In the limit of vanishing information frictions, our proposed program achieves the first-best outcome at zero cost. The results are robust to several generalizations including, agents' unobservable ex-ante heterogeneity, continuous payoff structure and a finite number of agents.

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1 Introduction

In coordination games, strategic complementarities among agents can give rise to multiple equilibria, some more efficient than others in terms of social welfare. In these games, coordination failure, that agents play an inefficient equilibrium, can leads to huge welfare loss. Therefore, the policy maker may find it desirable to intervene and incentivize the agents to play the efficient equilibrium. For example, some governments provide demand deposit insurance to avoid panic-based bank runs.[1] However, many existing intervention policies subsidize all or a certain group of agents uniformly. These policies suffer from two drawbacks. First, they are costly to implement due to their large size. Back to the deposit insurance example, although theoretically government guarantees are costless, in practice, the administrative costs and the opportunity costs of maintaining the deposit insurance fund are enormous.[2] Even for "divide-and-concur" programs that target at a certain group of agents, when agents have heterogeneous private information, these policies might be wasting resources on the optimistic agents who would have taken the efficient action without the policies. Second, these policies might induce additional welfare loss due to moral hazard problems. For example, demand deposit insurance has been criticized for inducing excessive risk taking by banks (Kareken and Wallace, 1978; Keeley, 1990). In this paper, we propose a type of intervention program that screens agents based on their interim beliefs. In equilibrium, only a small group of "pivotal" agents self-select to participate in the program, which reduces the implementation costs. Moreover, moral hazard problems are only limited to the participating agents.

In our benchmark model, we explore a canonical binary-action coordination game under the standard global games framework. Global games are useful for linking coordination outcome to the underlying fundamental and determining the unique equilibrium. More importantly, they highlight the strategic interactions of agents with heterogeneous private information. Specifically, in our model, each agent is endowed with an investment opportunity. The investments would be successful if and only if the fraction of agents investing exceeds certain threshold which decreases in the fundamental of the economy. In addition, each agent receives a noisy private signal of the fundamental and makes inferences about the other agents' investment decisions. The game has a unique threshold equilibrium where all agents follow the same threshold strategy. In terms of welfare, there exists a region of weak fundamentals in which agents walk away from their investment opportunities, however,

[1]See Rochet and Vives (2004), Corsetti et al. (2006) and Morris and Shin (2006) for policies of deposit insurance or "lender of last resort" to prevent panic-based runs on financial institutions

[2]The operating expenses of deposit insurance fund has been increasing over time to over $1.7 billion in 2016. See Federal Deposit Insurance Corporation, 2016 Annual Report.

the investments would have been successful if all agents were to invest. Therefore, social welfare will be improved if the policy maker can lower the investment threshold and reduce the coordination failure region.

Next, we allow the policy maker to offer a stimulus program with voluntary participation to all agents who invest. If an investor accepts the offer, she receives a direct subsidy. In return, she is required to pay tax when the investment is successful. We classified the stimulus programs into three categories based on the subsidy-to-tax ratio. If a program is too austere, i.e. low subsidy-to-tax ratio, no agents will participate. If a program is too generous such that all investors will participate, we call it a **full-participation program**. As mentioned before, many existing intervention policies are uniformly beneficial to all agents, therefore, fall into this category. We show that full-participation programs can effectively reduce coordination failure however are costly to implement. What we propose is **partial-participation programs** with medium subsidy-to-tax ratios. A partial participation program is equivalent to a costly insurance policy so it screens agents based on their interim belief of success. The most optimistic investors who believe in a high probability of paying the tax don't take the offer. At the same time, the most pessimistic agents won't invest solely to take advantage of the offer since they still have "skin in the game". Only agents with intermediate belief will participate in the program since it partially protect them from investment loss. We show that with a partial-participation program there is a unique Bayesian Nash equilibrium, in which all agents follow the same threshold strategy with one investment threshold and one participation threshold. The investment threshold is lower than the participation threshold. An agent invests if and only if her private signal is above the investment threshold, but only participates in the program if the signal is between the two thresholds. When the information friction goes to zero, the two thresholds converge, and the expected mass of agents who accept the offer goes to zero, which implies zero expected cost of implementation for the policy maker. Furthermore, with proper choice of subsidy and tax, coordination failures can be eliminated, and the first best investment threshold can be achieved.

To understand intuitively how partial-participation programs can improve coordination results at minimal cost, let's start with the original threshold equilibrium without stimulus programs. For agents receiving signals right below the investment threshold, with the extra incentive provided by the offer, they would be willing to change their choice and invest. Since all agents rationally expect more investors, strategic complementarities strengthen all agents' incentive to invest. Hence, agents receiving even lower signals would be willing to accept the offer and invest. This effect is amplified by higher-order beliefs and lowers the investment threshold significantly. Now that all agents are more optimistic, the offer becomes less appealing, and the mass of investors who accept the offer in equilibrium is actually small.

We then compare full-participation programs and partial-participation programs in the presence of moral hazard problem. We extend the benchmark model by assuming after investment, investors can earn private benefit by shirking which decreases the investment success probability. The stimulus programs reduce investors' "skin in the game" hence induce shirking at the expense of the policy maker and social welfare. Moral hazard problem critically limits the magnitude of the full-participation programs. Specifically, if a full-participation program offers large direct subsidy and tax, all investors would participate and shirk. In contrast, for partial-participation programs, only the "pivotal" agents self-select to participate and shirk. For the optimistic agents, rejecting the offer and exerting effort gives higher payoff than participating and shirking. Hence, the social welfare loss only incurs for medium-belief agents, the mass of whom goes to zero in the limit of vanishing information frictions. As a result, in the limit, there still exist partial-participation programs that can restore the first-best, yet no full-participation programs can restore the first-best.

Besides the benchmark model, we also show that the results could be generalized to unobservable ex-ante agent heterogeneity, a continuous payoff structure and a finite number of agents. Regarding ex-ante agent heterogeneity, a closely related paper is Sakovics and Steiner (2012). The difference is that they only allow the policy maker to provide subsidies conditional on agents' observable heterogeneities. Under their set up, the most cost-efficient subsidies should target at agents with specific ex-ante characteristics. However, their policy space falls into the category of full-participation programs in our model, and the policy maker can save costs and limit moral hazard problems by switching to a partial-participation program. In other words, we show that subsidization should target the interim rather than ex-ante "pivotal" types. Moreover, since the "pivotal" agents self-select to participate in partial-participation programs, the policy maker doesn't need to observe agents' ex-ante characteristics.

Our paper is related to three lines of literature. First, our model is built on the literature on global game techniques that resolves multiple equilibria in games with strategic complementarities. The global games literature has been pioneered by Carlsson and Van Damme (1993) and Morris and Shin (1998), and the most commonly applied setup and applications are reviewed in Morris and Shin (2003). Our main model in section 2 is a special case with binary payoffs. In section 3.3, we discuss a generalized payoff structure as in Morris and Shin (2003). In both cases, we show that there exists costless intervention to reduce the coordination threshold and eliminate coordination failures in the limit of zero information friction.

Second, our paper is connected to the literature that applies global game techniques and evaluates policies/interventions to reduce coordination failures. Among the various applications, the most prominent ones are but not limited to bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005), credit freeze (Bebchuk and Goldstein, 2011), debt rollovers (Morris and Shin, 2004a; He and Xiong, 2012), liquidity run in financial markets (Bernardo and Welch, 2004; Morris and Shin, 2004b), investment crashes (Chamley, 1999; Dasgupta, 2007) and political revolution (Edmond, 2013). Other applications include underinvestment, underemployment in macroeconomics (Cooper and John, 1988), industrial organization (Dasgupta, 2007) and political revolutions (Edmond, 2013). Unlike these papers, we don't focus on the application under a specific context, but rather make policy implications that fit into different contexts. Bebchuk and Goldstein (2011) discussed various policies to reduce coordination failures leading to credit freeze. Most of the policies discussed in the literature would incur pecuniary costs for the government or create incentive problems for the participating agents.

Third, our mechanism shares similar ideas found in the literature that explores policies targeting at a specific group of agents to reduce coordination failures. For example, within the contracting literature, Segal (2003) and Bernstein and Winter (2012) show that the optimal policy is to *divide and conquer*, i.e. subsidize a subset of players so that they invest even if no one else in the complement set invests, then the surplus of players in the no-subsidy set are fully extracted. Sakovics and Steiner (2012) and Choi (2014) analyzed a coordination game with ex-ante heterogeneous agents and showed that different types should be subsidized in a certain order. These papers all demonstrate that subsidizing a subset of agents to ensure their participation can efficiently encourage the participation of the rest of agents and reduce coordination failure. Our proposed intervention program is different in terms of implementation. The policy maker offers the same option to all agents, and a subset of agents self-select to participate in the program. In the generalization of unobservable ex-ante heterogeneity, we show that our proposed stimulus program is more efficient and doesn't require information about agents' heterogeneity. Another closely related paper is Morris and Shadmehr (2017), which analyzes the reward schemes for a revolutionary leader to elicit efforts from citizens. The optimal reward scheme also screens citizens for their optimism. However, they consider bounded reward schemes imposed on a continuous and unbounded effort choice set, while we focus on subsidy-tax programs that agents can voluntarily choose to participate in. More importantly, while they assume zero cost for implementing any reward scheme, we target at minimizing the cost of intervention.

The rest of the paper is organized as follows. We present a benchmark model of an investment game with simple payoff structure in Section 2 and demonstrate the mechanism of the proposed partial-participation program. Section 3 compares the proposed program with other programs such as the lender of last resort in the presence of moral hazard problems. Several extensions of the benchmark model are discussed in Section 4. Section 5 presents several applications of the benchmark model and discusses policy recommendations in each context. Finally, section 6 concludes.

2 The Benchmark Model

In this section, we discuss a benchmark investment game with coordination failure in a global game setup. Then we show how the proposed stimulus method can encourage investment and achieve the socially optimal outcome. The size and cost of such stimulus programs go to zero when the information friction vanishes.

2.1 Setups

There is a unit mass of ex-ante identical infinitesimal agents, indexed by $i \in [0, 1]$. These agents are endowed with the same investment opportunity. And they simultaneously make investment decisions $a_i \in \{0, 1\}$. $a_i = 1$ if agent i decides to invest, and $a_i = 0$ if agent i walks away from the investment opportunity. Walking away results in zero payoffs, while investing incurs a fixed cost $c > 0$ and generates a profit of $b > c$ if agent i's project is successful and 0 if it fails. We assume all agents' investment payoffs are perfectly correlated. The investments would be successful when the fundamentals of the economy are strong enough and/or a sufficient number of agents invest. Specifically, the payoff from an investment project is

$$\pi(\theta, l) = \begin{cases} b - c, & \text{if } l \geq 1 - \theta, \\ -c, & \text{if } l < 1 - \theta. \end{cases}$$

where $l = \int_0^1 a_i \mathrm{d}i$ represents the fraction of investors or the aggregate investment level, and θ stands for the fundamentals of the economy. Note that agents' investment decisions feature strategic complementarities, because each project is more likely to succeed when more agents choose to invest. When the fundamentals are higher, it requires less aggregate investment to make the projects successful. Without information friction, when $\theta \in [0, 1)$, all agents investing ($l = 1$) and all agents walking away ($l = 0$) are both Nash equilibria. However, all agents investing is strictly more efficient than the other equilibrium. Therefore, the first-best outcome is that all agents coordinate to invest when $\theta \geq 0$ and walk away when $\theta < 0$.

We follow the standard global game set up and assume the following information structure. Without loss of generality, assume the fundamental θ is drawn from an uniform distribution with support $[\underline{\theta}, \bar{\theta}]$. The fundamental is not directly observable to the agents when they make investment decisions. Instead, each agent receives a noisy signal about the fundamental $x_i = \theta + \sigma\varepsilon_i$, where ε_i is identically and independently distributed with a continuous and strictly increasing c.d.f. $F(\epsilon)$, the support of which is $[-\frac{1}{2}, \frac{1}{2}]$. Furthermore, we assume that $\underline{\theta} < -\sigma$ and $\bar{\theta} > 1 + \sigma$. Under this assumption, there exist two dominance regions of signals, $[-\underline{\theta} - \frac{1}{2}\sigma, \underline{x})$ and $(\bar{x}, \bar{\theta} + \frac{1}{2}\sigma]$, with $\underline{x}$ and $\bar{x}$ defined as

$$\Pr(\theta \geq 1 | x = \bar{x}) = \frac{c}{b},$$
$$\Pr(\theta \geq 0 | x = \underline{x}) = \frac{c}{b}.$$

Intuitively, with the lowest aggregate investment level $l = 0$, an agent is indifferent between the two actions when she receives signal $\bar{x}$. Therefore, her dominant strategy when signal $x > \bar{x}$ is to invest. Similarly, with the highest aggregate investment level $l = 1$, an agent is indifferent between the two actions if she observes signal $\underline{x}$. Hence, when $x < \underline{x}$, walking away is the dominant strategy.

2.2 Equilibrium without Intervention

In this subsection, we analyze the equilibrium without intervention and identify the inefficiencies due to coordination failure. Proposition 1 characterizes the equilibrium.

Proposition 1 *Without intervention, there is a unique equilibrium in which all agents follow the same strategy*

$$a_i(x_i) = \begin{cases} 1, & \text{if } x_i \geq \xi_0^*, \\ 0, & \text{if } x_i < \xi_0^*. \end{cases}$$

where $\xi_0^* = \frac{c}{b} + \sigma F^{-1}\left(\frac{c}{b}\right)$.

Since there is a continuum of agents, given the realization of fundamentals θ, we can apply the law of large numbers to calculate the aggregate investment l and predict the coordination outcomes. In equilibrium, all agents follow the same threshold strategy. Therefore, the coordination outcome also has a threshold above which the investment projects are successful. The fundamental threshold is given by

$$\theta^*(\xi_0^*) = \frac{c}{b}.$$

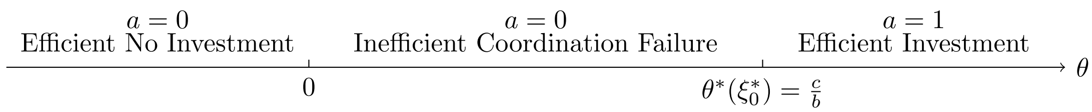


Figure 1: Coordination Results

The fundamental realizations can be divided into three regions as shown below. In the middle region $\theta \in \left[0, \frac{c}{b}\right)$, if all agents coordinate to invest, the investment projects would have been successful. However, the agents have self-fulfilling beliefs that other agents will walk away from their investment opportunities. As a result, they rationally choose not to invest. Since a unit of successful investment generates a positive surplus of $b - c$, in the middle region, coordination failure leads to social welfare loss of $b - c$. Hence, the first-best scenario has a fundamental threshold θ^* equal to zero. And in the next section, we will show how our proposed stimulus program can lower this cutoff and reduce inefficiencies caused by coordination failure.

2.3 Stimulus Program

Having characterized the equilibrium in the game without intervention, we now describe the subsidy-tax stimulus program that the policy maker can use to boost investment and reduce coordination failure. The stimulus program consists of two parts, a direct subsidy $s \in [0, c]$ and a contingent tax $t \in [0, b)$. Specifically, if an investor decides to accept the stimulus offer, she receives an upfront subsidy s regardless of the investment outcome and pays a lump-sum tax t only if the investment succeeds.[3] The program is only available to investors and they voluntarily decide whether to participate in the program. Note that there is an implicit assumption that the actions taken by the agents are observable to the policy maker and can be contracted on. We make this assumption because, as shown in Bond and Pande (2007), if the policy maker cannot observe individual actions, its ability to use subsidy-tax schemes as a coordination device is greatly limited. This assumption imposes certain limitations on the application of our proposed intervention mechanism. For example, in context of currency attack, it is hard to trace agents' action and tax conditional on agents' investment behaviour. Therefore, the stimulus program discussed in this paper cannot be applied to solving currency deflation caused by coordination failure (see Morris and Shin (1998)). Despite this limitation, there are a wide range of real-world applications. In section

[3]Since in the benchmark model there's only two possible payoffs from investing, we only need to specify a contingent lump-sum tax. In section 4.2, we analyze a more general setup where there's a continuum of investment outcomes and we allow tax to be proportional to the investment revenue.

5, we discuss three representative examples.

Mathematically, if an investor accepts the offer, her payoff is modified to

$$\tilde{\pi}(\theta, l) = \begin{cases} b - t - (c - s), & \text{if } l \geq 1 - \theta, \\ -(c - s), & \text{if } l < 1 - \theta. \end{cases}$$

The upfront payment s reduces the cost of investment and encourages agents to invest. The contingent tax t directly helps the policy maker recover the cost of providing subsidies. More importantly, it will become clear later that the contingent tax t indirectly saves cost by deterring participation of optimistic agents. The timeline of the coordination game with the stimulus program is modified as follows. At the beginning of the game, the policy maker announces the stimulus program (s, t). Then the fundamental θ is realized, and each agent receives a noisy signal of the fundamental. After observing the signal, agents simultaneously make their decisions on whether to invest and if invest, whether to participate in the stimulus program. As soon as the decisions are made, active investors pay the cost c, and the policy maker transfers the subsidy s to all investors participating in the stimulus program. Then the fundamental θ and the investment returns are realized. Finally, the policy maker collects tax t from the investors participating in the stimulus program if the investments are successful. The timeline is summarized in figure 2.

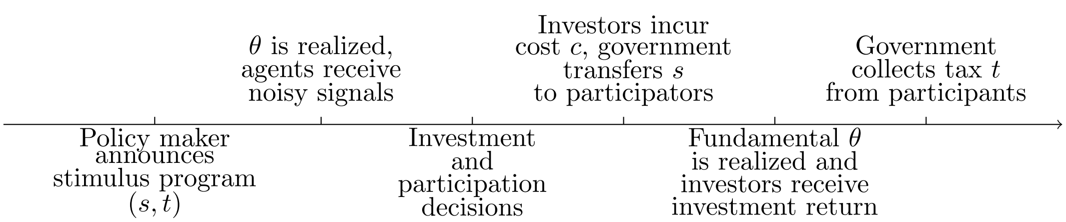


Figure 2: Timeline of the investment game

Although the intervention program is specified as a subsidy-tax program, it can be interpreted as other forms of intervention with transfers between the policy maker and the investors contingent on the coordination result. For example, a government-guarantee type of program that promises to cover the loss of failed investment up to $s^g \leq c$ is equivalent to a subsidy-tax program with $s = t = s^g$. To see this, under both programs, the net transfer from the government to any participating investor is 0 in the case of successful investments and s^g in the case of failed investments. Similarly, an asset purchase program in which the policy maker buy $\frac{t}{b}$ fraction of the project with price s is equivalent to a subsidy-tax program (s, t).

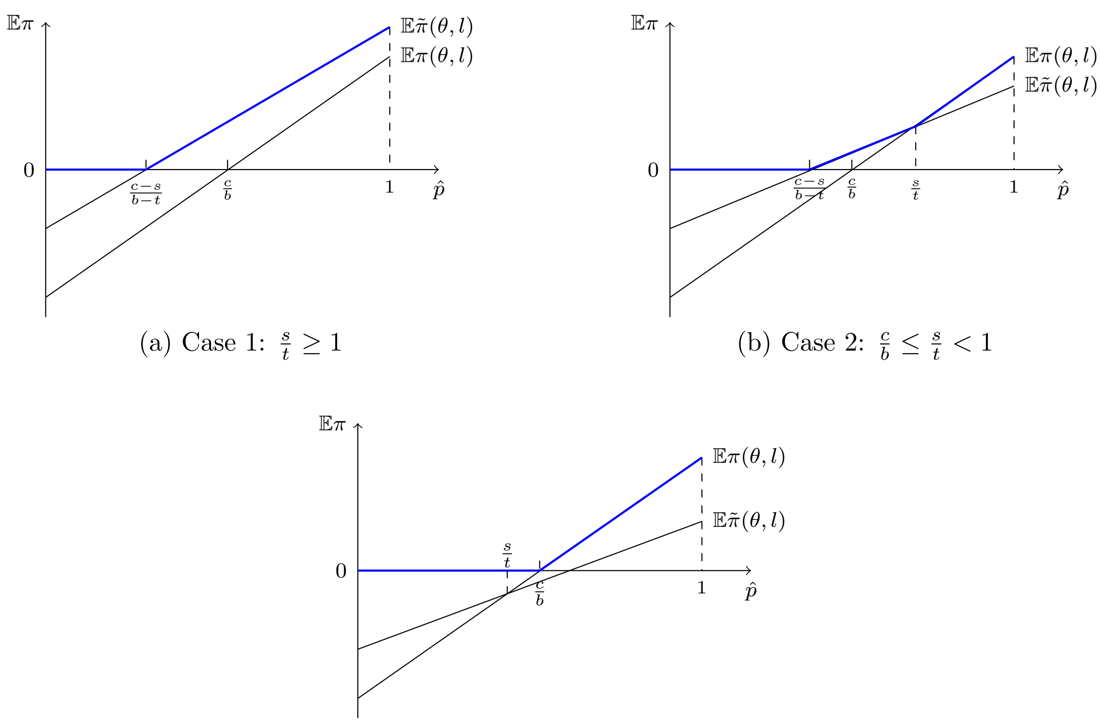
2.4 Equilibrium with Intervention

We now analyze the equilibrium with intervention and demonstrate how the stimulus program works to reduce coordination failure. With the stimulus program, an agent has three choices: $\{a = 1, \mathit{Reject}\}$, $\{a = 1, \mathit{Accept}\}$ and $\{a = 0\}$. Note that although agents make two decisions, whether to invest and conditional on investing, whether to accept the offer, only their investment decisions affect the coordination results. Therefore, an agent only cares about the investment decisions of the others but not their participation in the stimulus program. As a result, to analyze the best response and equilibrium strategies, it is sufficient to condition on other agents' investment strategies. Let $\hat{p}_i = \Pr[l \geq 1 - \theta | x_i, a_{-i}(x)]$ be the posterior belief of success of agent i given her private signal x_i and other agents' investment strategies $a_{-i}(x)$. The expected payoffs from $\{a = 1, \mathit{Reject}\}$ and $\{a = 1, \mathit{Accept}\}$ are

$$\mathbb{E}\pi(\theta, l) = \hat{p}_i b - c, \tag{1}$$

$$\mathbb{E}\tilde{\pi}(\theta, l) = \hat{p}_i (b - t) - (c - s) \tag{2}$$

respectively. And the expected payoff from $\{a = 0\}$ is zero. Figure 3 depicts the expected payoff as a function of the posterior belief $\hat{p}$. It can be divided into three cases according to the subsidy-tax ratio $\frac{s}{t}$. In the first case with $\frac{s}{t} \geq 1$, accepting the stimulus offer dominates



(a) Case 1: $\frac{s}{t} \geq 1$

(b) Case 2: $\frac{c}{b} \leq \frac{s}{t} < 1$

(c) Case 3: $\frac{s}{t} < \frac{c}{b}$

Figure 3: Expected payoffs and posterior beliefs.

rejecting the offer. This is because investors always receives a higher subsidy s than their tax payment required by the stimulus program. We call this type of programs the full-participation programs. Without intervention, the belief threshold for investment is the cost-benefit ratio $\frac{c}{b}$. With a full-participation program, the threshold is lowered to $\frac{c-s}{b-t}$. In the third case with $\frac{s}{t} < \frac{c}{b}$, rejecting dominates accepting the offer. We call this type of programs the zero-participation programs. Thus, the threshold belief under the zero-participation program is the same as the original cost-benefit ratio $\frac{c}{b}$. The second case is the most interesting one. When $\frac{c}{b} \leq \frac{s}{t} < 1$ (figure 3.b), an agent would only accept the offer and invest when she has an intermediate belief $\hat{p} \in [\frac{c-s}{b-t}, \frac{s}{t}]$. We call this type of programs the partial-participation programs. Notice in both case 1 and case 2, the provision of the stimulus program lowers the threshold belief to $\frac{c-s}{b-t}$. The difference is that, in case 2, the most optimistic agents don't participate in the stimulus program, which is cost saving especially when the information friction is small. We will analyze the cost of the programs in detail in section 2.5.

Next we sketch the analyses of equilibrium with intervention. It will become clear later that iterated deletion of dominated strategies allows us to focus on cutoff investment strategies. We say an agent follows a cutoff investment strategy with threshold k, if her investment strategy is

$$a_i(x;k) = \begin{cases} 1, & \text{if } x \geq k, \\ 0, & \text{if } x < k. \end{cases} \tag{3}$$

Let $p(x;k)$ denote the posterior belief of success when an agent receives private signal x and all other agents follow a cutoff investment strategy k,

$$p(x;k) = Pr(\theta > \theta^*(k)|x) = F\left(\frac{x - \theta^*(k)}{\sigma}\right), \tag{4}$$

where $\theta^*(k)$ is fundamental threshold for successful investment and satisfies $F\left(\frac{k-\theta^*(k)}{\sigma}\right) = \theta^*(k)$. An agent's posterior belief of success $p(x;k)$ increases in x and decreases in k, because a high private signal x indicates a high realization of fundamentals θ, and a low investment threshold k implies a high aggregate investment l. Both implies a high probability of success.

In all three cases depicted in figure 3, the optimal investment strategy is an agent invests if and only if her belief $p(x,k)$ exceeds a threshold. Since $p(x,k)$ is monotonic in both x and k, an agent's best response to other agents' cutoff strategy k is also a cutoff investment strategy based on her own signal. The two dominance regions form two extreme cutoff investment strategies. Starting there, by iterated deletion of dominated strategies, we are able to prove the uniqueness of the equilibrium with intervention. The details of the analyzes can be found in the proof of proposition 2 below. The following proposition characterizes the equilibrium with a subsidy-tax stimulus program (s, t).

Proposition 2 *When the policy maker offers a subsidy-tax stimulus program $(s, t) \gg 0$, the game has a unique equilibrium. There are three different cases:*

1. *When $\frac{s}{t} \geq 1$, the equilibrium is for any agent i,*

$$\begin{aligned} a_i = 1, \textit{Accept}, \quad \textit{if} \ \ x_i \geq \xi^*(s,t), \\ a_i = 0, \quad \textit{if} \ \ x_i < \xi^*(s,t). \end{aligned}$$

   where

$$\xi^*(s,t) = \frac{c-s}{b-t} + \sigma F^{-1}\left(\frac{c-s}{b-t}\right),$$

2. *When $\frac{c}{b} \leq \frac{s}{t} < 1$, the equilibrium is for any agent i,*

$$\begin{aligned} a_i = 1, \textit{Reject}, \quad \textit{if} \ \ x_i \geq \eta^*(s,t), \\ a_i = 1, \textit{Accept}, \quad \textit{if} \ \ \xi^*(s,t) \leq x_i < \eta^*(s,t), \\ a_i = 0, \quad \textit{if} \ \ x_i < \xi^*(s,t), \end{aligned}$$

   where

$$\begin{aligned} \xi^*(s,t) = \frac{c-s}{b-t} + \sigma F^{-1}\left(\frac{c-s}{b-t}\right), \\ \eta^*(s,t) = \frac{c-s}{b-t} + \sigma F^{-1}\left(\frac{s}{t}\right). \end{aligned}$$

3. *When $\frac{s}{t} < \frac{c}{b}$, the equilibrium is for any agent i,*

$$\begin{aligned} a_i = 1, \textit{Reject}, \quad \textit{if} \ \ x_i \geq \xi^*(s,t), \\ a_i = 0, \quad \textit{if} \ \ x_i < \xi^*(s,t), \end{aligned}$$

   where

$$\xi^*(s,t) = \frac{c}{b} + \sigma F^{-1}\left(\frac{c}{b}\right).$$

The ratio of the upfront transfer s and the ex-post tax t can be interpreted as the generosity of the stimulus offer. If the offer is generous (case 1), all investors find it profitable to accept the offer and the equilibrium investment cutoff depends on the modified cost $c' = c - s$ and benefit $b' = b - t$. If the offer is austere (case 3), all investors won't be interested in the offer. Therefore the equilibrium investment cutoff is the same as the original cutoff without stimulus offer. The most interesting case is case 2, in which the generosity of the offer is medium. Investors with high private signals have strong beliefs in the success of the project, so they will reject the subsidy offer since they believe in a high probability of paying a higher tax in the future. However, even without subsidies, these optimistic agents would invest anyway. Agents with low private signals have strong beliefs in the failure of the project, so even with the subsidy s, they still suffer a loss of $c - s$ from investing. Therefore, these agents would walk away from the investment opportunities with or without the stimulus program. In contrast, investors receiving signals around the threshold don't have strong beliefs about the coordination results. Without the stimulus program, some of these agents would not invest. The stimulus program provides insurance against losses in case of failed investment and gives these agents extra incentive to invest. With the extra incentive, these agents' decisions are effectively altered and the aggregate action l therefore increases. The increase in l, in turn, strengthens all agents' incentive to invest. Agents with even lower signals would accept the stimulus offer and change their decisions to invest. Through iterations of higher-order beliefs, the action cutoff is significantly lowered. Moreover, agents with signals around the old cutoff are significantly more optimistic, and therefore the stimulus package is no longer appealing to them. In equilibrium, the mass of investors accepting the offer is rather small. We call these investors the "pivotal" investors, since the equilibrium investment cutoff is determined by their modified cost and benefit.

In case 1 and 2, the fundamental cutoff above which the investment projects are successful is

$$\theta^*(\xi^*(s,t)) = \frac{c-s}{b-t}. \tag{5}$$

Note that the new fundamental cutoff is lower than that without government stimulation. Therefore, the provision of the stimulus program successfully reduces the inefficient coordination failure region. If the government picks $s = c$ and $t \in [s, b)$, the fundamental cutoff can be reduced to 0, eliminating the whole region of inefficient coordination failure as demonstrated in Figure 4.

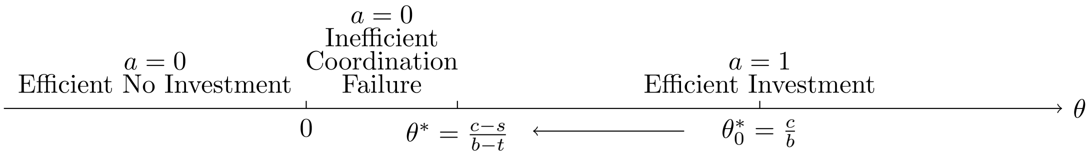


Figure 4: Coordination Results after Intervention

2.5 Cost of the Stimulus Program

In this section, we discuss the cost efficiency of different subsidy-tax stimulus programs. In particular, we compare the expected cost of the partial-participation and full-participation programs conditional on the same target fundamental threshold θ^* of successful investment. To allow for the possibility that the policy maker values tax and subsidy differently, the value of tax for the policy maker is normalized to 1 and the cost of providing subsidy is assumed to be τ. The ex-post cost of providing the stimulus program to an individual investor is

$$\hat{c}(\theta, s, t) = \begin{cases} \tau s - t, & \text{if } l \geq 1 - \theta, \\ \tau s, & \text{if } l < 1 - \theta. \end{cases} \tag{6}$$

When $\hat{c}(\theta, s, t)$ is negative, the policy maker profits from providing this stimulus program.

For the rest of the analyses, we focus on $\tau \geq 1$ for two reasons. First, we believe it is a realistic characterization. If subsidy is provided before tax collection, $\tau > 1$ reflects the funding cost of the policy maker due to the opportunity cost of other welfare improving programs. Alternatively, if the program is government guarantee, $\tau > 1$ reflects the cost of commitment, such as setting aside funds specifically for the program. Moreover, any administrative cost incurred by providing subsidy or collecting tax can increase τ. Secondly, if $\tau < 1$, given negligible information frictions, the policy maker can easily restore the first best and profit at the same time by offering $t = s = c$.[4] The coordination problem then becomes trivial. Therefore, for the rest of the paper, we assume $\tau \geq 1$.

Let $C(\theta, s, t; \sigma)$ denote the ex-post total cost of providing a subsidy-tax stimulus program (s, t) given the realized fundamental θ and the information friction σ. For full-participation programs, i.e. $\frac{s}{t} \geq 1$, all investors participate in the stimulus program. The ex-post cost of

[4]To be precise, the policy maker should set $t = s = c - \varepsilon$ with a very small ε to avoid over-investment when $\theta < 0$.

implementation is

$$C(\theta, s, t; \sigma) = \begin{cases} (\tau s - t)\left[1 - F\left(\frac{\xi^*(s,t)-\theta}{\sigma}\right)\right], & \text{if } \theta \geq \frac{c-s}{b-t}, \\ \tau s\left[1 - F\left(\frac{\xi^*(s,t)-\theta}{\sigma}\right)\right], & \text{if } \theta < \frac{c-s}{b-t}. \end{cases} \tag{7}$$

For partial-participation programs $\frac{c}{b} \leq \frac{s}{t} < 1$, only pivotal investors participate in the stimulus program. In this case,

$$C(\theta, s, t; \sigma) = \begin{cases} (\tau s - t)\left[F\left(\frac{\eta^*(s,t)-\theta}{\sigma}\right) - F\left(\frac{\xi^*(s,t)-\theta}{\sigma}\right)\right], & \theta \geq \frac{c-s}{b-t}, \\ \tau s\left[F\left(\frac{\eta^*(s,t)-\theta}{\sigma}\right) - F\left(\frac{\xi^*(s,t)-\theta}{\sigma}\right)\right], & \theta < \frac{c-s}{b-t}. \end{cases} \tag{8}$$

If $\frac{s}{t} < \frac{c}{b}$, no agents will find it profitable to opt in the stimulus program, therefore $C(\theta, s, t; \sigma) = 0$.

Proposition 3 below compares the ex-post and ex-ante expected cost of partial-participation programs and full-participation programs that restores first best in the limit of vanishing information frictions.

Proposition 3 *With strictly costly subsidy $\tau > 1$, when the information friction σ goes to 0, there exists a continuum of full-participation programs (s, t) and a continuum of partial-participation programs (s', t') achieving the first-best outcome, where $s = s' = c$ and $t \leq c < t' \leq b$.*

For any such (s, t) and (s', t'), given θ, the full-participation program (s, t) is ex-post more costly than the partial-participation program (s', t'). Specifically,

$$\begin{aligned} \lim_{\sigma \to 0} C(\theta, s, t; \sigma) = \tau s - t > \lim_{\sigma \to 0} C(\theta, s', t'; \sigma) &= 0, && \text{if } \theta > 0; \\ \lim_{\sigma \to 0} C(\theta, s, t; \sigma) = \tau s - t > \lim_{\sigma \to 0} C(\theta, s', t'; \sigma) &= \tfrac{s'}{t'}(\tau s' - t'), && \text{if } \theta = 0; \\ \lim_{\sigma \to 0} C(\theta, s, t; \sigma) = \lim_{\sigma \to 0} C(\theta, s', t'; \sigma) &= 0, && \text{if } \theta < 0. \end{aligned}$$

Moreover, the full-participation program (s, t) is ex-ante strictly more costly than the partial-participation program (s', t'). Specifically, $\lim_{\sigma \to 0} \mathbb{E}_\theta C(\theta, s, t; \sigma) > \lim_{\sigma \to 0} \mathbb{E}_\theta C(\theta, s', t'; \sigma) = 0$.

The proof is in the Appendices. When the information friction is small, although both full-participation programs and partial-participation programs can effectively reduce coordination failures and restore the first-best outcome, the partial-participation programs are ex-post weakly less costly than the full-participation programs in all states. Intuitively, compared with full-participation programs, partial-participation programs significantly reduce the size of the program by deterring the optimistic investors from participating the program.

This subsequently reduces the cost of implementation. If the policy maker evaluates the ex-ante expected cost of the programs, in the limit of negligible information frictions, the partial-participation programs incur zero cost and strictly dominate the full-participation programs.

Now we extend the analyses to the case of non-negligible information frictions $\sigma > 0$. To facilitate the comparison of full-participation programs and partial-participation programs, we introduce an alternative parameterization of stimulus programs. Specifically, a stimulus program (s, t) can be equivalently parameterized by (θ^*, λ), where $\theta^* = \frac{c-s}{b-t}$ is the target fundamental threshold and λ represents the size of the stimulus program. Mathematically,

$$
\begin{aligned}
s &= \frac{c - \theta^* b}{1 - \theta^*} + \theta^* \lambda, \\
t &= \frac{c - \theta^* b}{1 - \theta^*} + \lambda.
\end{aligned}
$$

The new parameterization is useful because we will compare the cost of implementing stimulus programs that target at the same fundamental threshold θ^*. When $\lambda \in [-\frac{c-\theta^* b}{1-\theta^*}, 0]$, the program is a full-participation program, and when $\lambda \in (0, \frac{b-c}{1-\theta^*})$, the program is a partial-participation program. Note that, to achieve the same target θ^*, the partial-participation programs have larger size λ, meaning both higher subsidy and higher tax charge, than the full-participation programs.

Suppose the policy maker is considering switching from a full-participation program (θ^*, λ) to a partial-participation program (θ^*, λ'). The change in the cost of implementation comes from both the extensive and the intensive margin. On the extensive margin, the most optimistic investors will no longer enter the program. Hence, this effect always reduces the expected cost of intervention. However, on the intensive margin, the cost of providing the program to an individual investor could increase or decrease. Formally, the difference in unit cost is

$$
\hat{c}(\theta, s', t') - \hat{c}(\theta, s, t) = \begin{cases} (\tau\theta^* - 1)(\lambda' - \lambda), & \text{if } \theta \geq \theta^*, \\ \tau\theta^*(\lambda' - \lambda), & \text{if } \theta < \theta^*. \end{cases}
$$

With vanishing information frictions, the effect on the intensive margin is negligible because the mass of participants in partial-participation programs goes to zero except for the knife-edge case of $\theta = 0$. Therefore, switching to any partial-participation program will always reduce the cost of implementation. This is no longer true with non-negligible information frictions. In proposition 4, we provide two sufficient conditions such that switching to a partial-participation program reduces the expected cost of implementation.

Proposition 4 *With information friction* $\sigma > 0$*, if* $1 \leq \tau < G(\theta^*, 1)$ *or* $\theta^*(1+\sigma) < 1$*, a partial-participation program* $(s', t') = (\frac{c-\theta^* b}{1-\theta^*} + \theta^*\lambda, \frac{c-\theta^* b}{1-\theta^*} + \lambda)$ *with positive but small enough* λ *achieves* θ^* *at lower expected cost than any full-participation program targeting at* θ^**, where* $G(\alpha, \beta)$ *is defined for any* $0 \leq \alpha \leq \beta \leq 1$ *as*

$$G(\alpha, \beta) = \frac{\int_{F^{-1}(\alpha)}^{F^{-1}(\beta)} F(x)\mathrm{d}x}{\alpha(F^{-1}(\beta) - F^{-1}(\alpha))}.$$

The proof involves technical details and is left in the Appendix. Here we provide some intuitions. Since partial-participation programs provide more subsidy and charges more tax to each participants, the effect on the intensive margin depends on the ratio of expected mass of taxpayers to the expected mass of subsidy receivers. If $\tau < G(\theta^*, 1)$, the ratio is large enough such that the increase in expected tax collection is greater than the increase in expected subsidy provision. Hence, the effect on the intensive margin also works in favor of the partial-participation programs, and switching to a partial-participation program with small λ reduces the expected cost. Notice for any given $\theta^* < 1$, $G(\theta^*, 1) > 1$. Therefore, the special case of $\tau = 1$ always satisfies the first condition. The second condition governs the relative importance of the two margins. If θ^* and σ are jointly small, the participation threshold η^* for partial-participation programs is also small, therefore the mass of participants is significantly reduced. In particular, if the second condition holds, the effect on the extensive margin dominates that on the intensive margin, making the proposed partial-participation program less costly than any full participation programs. In summary, Proposition 4 gives three circumstances in which the most cost-efficient subsidy-tax program is a partial-participation program: ambitious target (small θ^*), small information frictions (small σ), or small cost of subsidy τ. One special case is when the policy maker targets at $\theta^* = 0$, the first best. In that case, there always exists a partial-participation program that dominates all full-participation programs.

We use a numerical example to demonstrates how switching to a partial-participation program from a full-participation program can reduce the expected cost of the intervention. Suppose the prior on θ is uniformly distributed on $(\underline{\theta}, \bar{\theta}) = (-0.2, 1.2)$. The private noise ε follows a uniform distribution over $[-\frac{1}{2}, \frac{1}{2}]$ and $\sigma = 0.2$. c and b are set to 1 and 1.25 so the benchmark success threshold is $\frac{c}{b} = 0.8$. The policy maker has a cost parameter $\tau = 1.05$ and targets at a success threshold $\theta^* = 0.2$. The least costly full-participation program to achieve the equilibrium threshold is $s = t = 0.9375$. The ex-post cost as a function of the realized fundamental is represented by the solid blue line in Figure 5. The cost is positive for all $\theta > \theta^*$ because all investing agents sign up for the program and there's a positive cost $\tau s - t$ of providing this program to each agent. When θ falls below θ^*, the cost surges because of the investment projects fail and the policy maker can't recover t. Now the policy maker switches to a partial-participation program. There's a continuum of partial-participation programs that targets at the same threshold θ^*. We take $(s', t') = (0.97, 1.1)$ for an example. The red line in the top panel of Figure 5 represents the ex-post cost function of program (s', t'). It has a similar shape as the cost function of the full-participation program. However, it converges to 0 when θ is large enough so that all agents receive signals higher than η^* and no agent participate in the stimulus program. The difference between the two cost functions is plotted in the bottom panel. Compared to the full-participation program, the partial participation program has lower cost when $\theta > \theta^*$ because of higher tax charge and less participation. When $\theta < \theta^*$, since the partial-participation program provides higher subsidy, it incurs higher cost than the full-participation program. On average, the partial-participation program incurs lower expected cost.

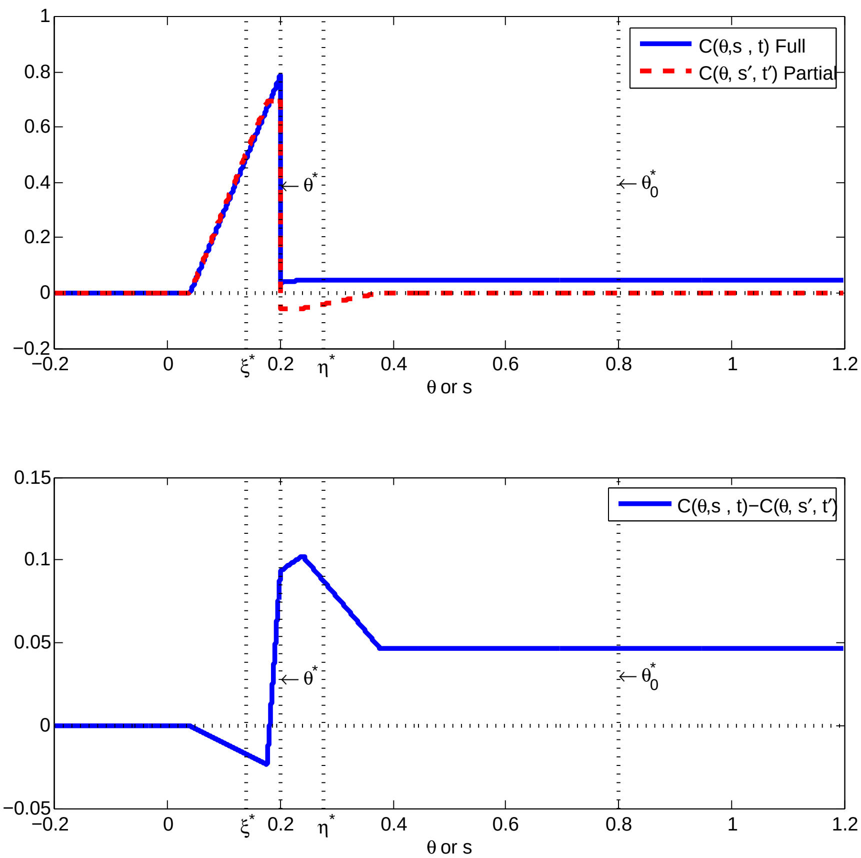


Figure 5: Cost Funtions

2.6 Discussions

From previous analyses, we show that partial-participation stimulus programs can improve the coordination results to the first-best outcome in the investment game, yet has zero cost when the information friction vanishes. This result seems striking at first glance. The most important reason why the partial stimulus program work effectively at a minimal cost is that it targets precisely the agents who are on the investment threshold and can be incentivized to invest relatively easily. These agents are also the "pivotal" investors whose investment decisions are crucial in the determination of the investment threshold. The figure below demonstrates how through higher-order beliefs our proposal effectively reduces coordination failure. ξ_0^* denotes the original cutoff without intervention. The partial

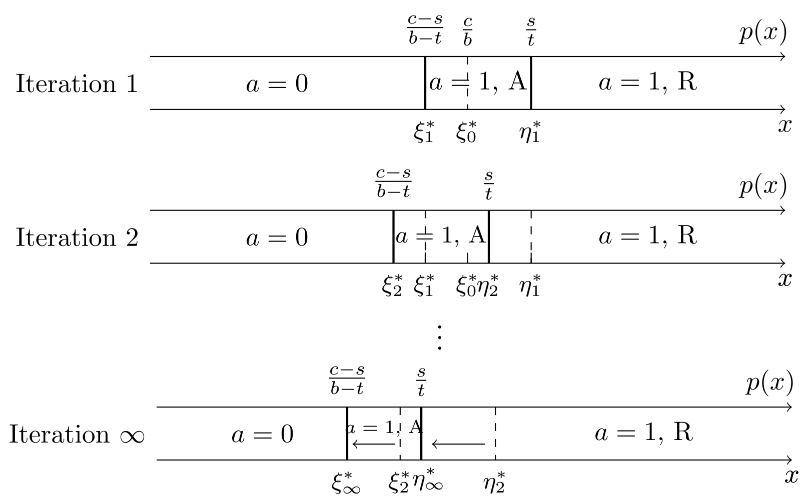


Figure 6: Role of Higher-Order Beliefs

stimulus program incentivizes agents to lower the investment threshold to ξ_1^*. Since all agents understand that more agents are willing to invest and therefore believe in a higher aggregate action l and a higher probability of successful investment, they are willing to lower their investment threshold further to ξ_2^*. Similarly, with the additional mass of agents receiving signals between ξ_1^* and ξ_2^* investing, all agents are more optimistic about the success of the investment and therefore further lower their investment threshold further to ξ_3. At the same time, as the agents become more optimistic about their investments, the stimulus program becomes less attractive, which implies a decreasing sequence of order thresholds η_n^*. With an infinite number of iterations, both investment threshold and order threshold are significantly lowered. As the information friction decreases, investors become more certain about the coordination results, so the mass of "pivotal" investors shrinks to zero. However, as long as there exist a few pivotal investors, the stimulus program will have a significant effect on the investment threshold due to higher-order beliefs.

It is clear that in the investment game presented above, full-participation programs are always dominated by partial-participation programs regarding cost. Take deposit insurance as an example. It has two important features. First, the participation is mandatory. FDIC charges a premium from banks for providing demand deposit insurance. Because of mandatory participation, the government can significantly lower the cost of such program. However, this may distort individual agents' incentive and welfare. Second, to credibly provide deposit insurance, the government need to spare certain budget aside to solve the commitment problem. Therefore, demand deposit insurance is not costless as in theory. To make a credible commitment, the government has to incur storage and opportunity costs. In Section IV, we discuss several applications of partial-participation programs and hope to shed light on policy designs in the future.

Our partial-participation programs share similar spirit to the targeted stimulus programs. Sakovics and Steiner (2012) analyzes coordination games with heterogeneous agents and argues that the optimal subsidy schedule is to target at a certain type of agents. In section 3, we examine an extension with heterogeneous agents and show that there exist partial-participation programs which incur zero cost to restore first best outcome in the limit of negligible information frictions. Similar to the main model, in equilibrium, only a small mass of "pivotal" agents self-select to accept the policy maker's offer. The only difference is that different types have different thresholds, and the "pivotal" agents are the ones receiving signals around their own thresholds. The result conveys one message contrasting Sakovics and Steiner (2012) that policy makers should target at interim rather than ex-ante important types. Also, one common problem with targeted stimulus programs is that information acquisition to identify the targeted type(s) can be costly. The policy maker needs to correctly identify each agent's type to implement the targeted stimulus programs. In contrast, our proposed stimulus programs incentivize the "pivotal" agents to self-reveal their types, therefore only require information on the payoff structure of different types. As a result, our proposed program is superior to the targeted stimulus programs in terms of saving the costs of collecting information.

3 Interventions in the Presence of Moral Hazard

The first extension is intended to demonstrate our proposal's robustness against moral hazard problem. In order to do that, we modify the game into two stages. The first stage is the same as the benchmark model with a stimulus program, except the payoffs are not realized until the second stage. If the realized fundamental $\theta < 1 - l$, we say the aggregate state is *Bad*. In this case, the game ends immediately, and all agents get zero payoffs. If the realized fundamental $\theta \geq 1-l$, we say the aggregate state is *Good*. In this case, the game enters the second stage, in which investors make their effort choices. If an investor exerts effort, the investor pays a cost of effort c^e, and his project succeeds with probability 1.[5] On the other hand, if an investor shirks, the project succeeds with probability $1-\gamma$. As in the benchmark model, the project pays b in case of success and 0 in case of failure. And for the participants in the stimulus program, they are required to pay tax t if their investments are successful. We make the following assumption on the parameters.

Assumption 1 *The project has the following properties:*

a) shirking is inefficient, $c^e < \gamma b$*;*

b) the investment projects are ex-ante efficient, $b > c + c^e$*.*

Given the assumptions above, the first-best scenario is that all agents invest and exert effort if the fundamental $\theta \geq 0$, and all agents walk away from their investment opportunities otherwise.

The equilibrium with moral hazard problem can be solved backward. In the second stage, an investor would exert effort if and only if

$$b - t - c^e \geq (1-\gamma)(b-t). \tag{9}$$

This condition can be interpreted as a constraint on the size of the tax t,

$$t \leq b - \frac{c^e}{\gamma}. \tag{10}$$

When the tax is above the threshold, participating investors will shirk, resulting in inefficient outcomes. Intuitively, with a higher costs of effort c^e or a lower losses caused by shirking γ, the incentive problem is more severe, imposing a tighter constraint on the size of tax t.

Next, we will analyze the equilibrium under different programs and examine whether a full-participation program like lender of last resort (LOLR) and a partial-participation program can achieve first best when there's moral hazard problem in the private investment project. In the context of our model, we interpret the LOLR program as a subsidy-tax program (s, t) with $s = t$, which is the full-participation programs with least cost. Since participating in the LOLR program weakly dominates investing alone, every investor will take advantage of this program.

[5]The results hold when the success probability when exerting effort is between $1-\gamma$ and 1.

Lender of Last Resort. The moral hazard problem in the second stage puts an upper limit on the size of the LOLR program if the policy maker wants to enforce effort.

The expected payoff from investing with the LOLR program is

$$U(x, A; k) = \begin{cases} p(x; k)(b - t - c^e) - (c - t), & \text{if } t \leq b - \frac{c^e}{\gamma}, \\ p(x; k)(1 - \gamma)(b - t) - (c - t), & \text{if } t > b - \frac{c^e}{\gamma}. \end{cases} \tag{11}$$

From the analysis of the benchmark model, we know that in the unique Bayesian Nash equilibrium, the fundamental threshold above which the aggregate state is good is equal to the belief of the marginal investor. Given a program with $t \leq b - c^e/\gamma$ that prevents shirking, the fundamental threshold in equilibrium is

$$\theta^* = \frac{c - t}{b - t - c^e}. \tag{12}$$

Given a program with $t > b - c^e/\gamma$ that tolerates shirking, the fundamental threshold in equilibrium is

$$\theta^* = \frac{c - t}{(1 - \gamma)(b - t)}. \tag{13}$$

In both cases, reducing the fundamental threshold to the first best $\theta^* = 0$ requires the subsidy s to be as close to c as possible. However, by the nature of the stimulus program, this also requires the contingent tax t to be as close to c as possible. The size of the stimulus program is constrained by the incentive constraint as shown in (10), and whether the constraint is binding depends on the severity of the moral hazard problem.

Assumption 2 *The moral hazard problem is severe,* $\frac{c^e}{\gamma} > b - c$.

Given Assumption 2 above, the maximum program size t that prevents shirking in the second period is strictly less than c, the cost of the investment project. Therefore, the LOLR program cannot achieve efficient fundamental threshold in the first stage and prevent shirking in the second period at the same time. The result is summarized in Proposition 5 below. When Assumption 2 doesn't hold, the LOLR program with $t \to c$ achieves the first best outcome.

Proposition 5 *Given Assumption 1 and 2, no LOLR program can restore the first best outcome when* $\sigma \to 0$.

Partial-participation Programs. Now let's consider a subsidy-tax program with $\frac{s}{t} \in [\frac{c}{b}, 1)$. Given that the tax is higher than the subsidy, whether to participate in the program depends on investors' idiosyncratic beliefs of the probability that the aggregate state is good. Intuitively, the program is more attractive to agents with intermediate beliefs as in the benchmark model. What complicates the analysis is the agents will take into account their shirking decisions in the second period when they compare the cost and benefit of participating in the program. When the moral hazard problem in the second period is not severe, i.e., Assumption 2 doesn't hold, the policy maker can choose $s \to c$ and $t = c$ to implement the first-best outcome, which is the same as LOLR programs. In the following analyses, we focus on the case when the moral hazard problem is severe and full-participation LOLR programs cannot achieve first best.

Given Assumption 2 holds and $t > c$, investors who participate the program will shirk, and investors who don't participate and invest will exert effort. The expected payoffs from different action choices are

$$U(x, A; k) = p(x; k)(1 - \gamma)(b - t) - (c - s), \tag{14}$$

$$U(x, R; k) = p(x; k)(b - c^e) - c, \tag{15}$$

$$U(x, 0; k) = 0. \tag{16}$$

The expected payoffs are linear and increasing in the belief $p(x; k)$ and the slopes are different. The difference in the slopes of $U(x, R; k)$ and $U(x, A; k)$,

$$(b - c^e) - (1 - \gamma)(b - t) = \gamma b + t(1 - \gamma) - c^e > 0 \tag{17}$$

is strictly positive given Assumption *1a*. Investing alone is the optimal choice if and only if the belief $p(x; k)$ exceeds the critical participation belief

$$p_2^*(s, t) \equiv \frac{s}{\gamma b + t(1 - \gamma) - c^e}. \tag{18}$$

Walking away from the investment opportunities is the optimal action choice if and only if the belief of the agent is worse than the critical investment belief

$$p_1^*(s, t) \equiv \frac{c - s}{(1 - \gamma)(b - t)}. \tag{19}$$

The optimal action choice if the belief of success probability is in between $p_1^*(s, t)$ and $p_2^*(s, t)$ is to invest and accept the stimulus offer.

Similar to those in the benchmark model, the critical beliefs determine the equilibrium cutoff of investment and participation regarding the private signal x. Investment efficiency in the first stage requires the critical investment belief $p_1^*(s, t)$ to be as close to 0 as possible.

It implies that the policy maker should choose subsidy $s = c$. On the other hand, if t can be selected properly such that the critical participation belief $p_2^*(s,t) < 1$, the investors who are very optimistic about the aggregate state would choose to invest and reject the stimulus offer. The exclusion of optimistic investors from the program improves efficiency in the second stage game and reduces the policy maker's cost from inefficient failures due to shirking. As the information friction goes to zero, only a zero mass of "pivotal" investors invest within the program. The following proposition summarizes the result.

Proposition 6 *Given Assumption 1 and 2, the equilibrium outcome given a subsidy-tax program* (s,t) *with* $s = c - \varepsilon$ *and* $\frac{c+c^e-\gamma b}{1-\gamma} < t < b$ *converges to the first best when* $\sigma \to 0$ *and* $\varepsilon \to 0$. *The ex-ante cost of policy maker of providing such program also converges to 0 when* $\sigma \to 0$.

Above proposition demonstrates the advantage of the partial-participation programs compared with full-participation programs like LOLR when the moral hazard problem is relatively severe. In the benchmark model, both types of programs can achieve the first best outcome at zero cost with diminishing information friction if $\tau = 1$. They are different in terms of the program size: full-participation programs include all investors, while partial-participation programs only target at the "pivotal" investors. Absent from other frictions, the size of the program doesn't alter the efficiency or the cost of the program. However, moral hazard problem imposes size-related inefficiency and cost on the program. When using a LOLR program, the policy maker faces a trade-off between the first stage investment efficiency and the second stage effort efficiency. A program with high subsidy over tax ratio ($\frac{s}{t}$) encourages investment in the first stage but deters effort input in the second stage. This trade-off limits the role of the LOLR program in improving social efficiency. On the contrary, despite the moral hazard problem, a partial-participation program still achieves the first best outcome at zero cost. The advantage of partial-participation programs in dealing with moral hazard problem is that they only involve a small mass of investors. Although these participating investors shirk in the second stage, it will have a limited impact on the social welfare since the size of these participating investors goes to zero at diminishing information friction. In General, the partial-participation program proposed in this paper is superior to the full-participation programs such as LOLR or government guarantee in the presence of any size-related inefficiency or cost.

4 Extensions

4.1 Unobservable Ex-ante Heterogeneity

In this part, we study whether the existence of ex-ante heterogeneity in agents' payoff structures and asymmetric information changes our results. The assumptions on the heterogeneity resemble those in Sakovics and Steiner (2012). Our analysis differs from their paper in two dimensions. First, they studied the optimal intervention when the policy maker can only make lump-sum subsidy, while we consider subsidy-tax programs. Second, they assume the types of agents are observable, while we allow for hidden types.

There are N groups of infinitesimal agents indexed by g, each group with mass m^g. There are three folds of heterogeneity. First, the agents differ in their agent profitability. They pay the same investment cost c yet earn different revenue b^g from successful investment. Assume there is no inefficient project, so $b^g > c$ for all g. Second, the agents impose different level of externalities for the coordination results. Specifically, the aggregate action $l = \sum_{g=1}^{N} \int_0^{m^g} w^g a_i^g di$. Same as in the benchmark model, the condition that investment is successful is $l \geq 1 - \theta$. The weights are normalized such that $\sum_{g=1}^{N} w^g m^g = 1$. Lastly, each agent receives a private signal $x_i^g = \theta + \sigma \varepsilon_i^g$, where ε_i^g is independent across agents and follows a group-specific distribution with c.d.f. $F^g(\varepsilon)$. We assume an agent's group is not observable to the policy maker. However, the policy maker knows the composition of agents.

The equilibrium without intervention is summarized by the following lemma.

Proposition 7 *Without intervention, there is a unique equilibrium in which an agent in group g invests if and only if her private signal is greater or equal to ξ_0^g,*

$$\xi_0^g = \sum_{g=1}^{N} m^g w^g \frac{c}{b^g} + \sigma F_g^{-1}\left(\frac{c}{b^g}\right). \tag{20}$$

From the above proposition, we can calculate the fundamental threshold above which investment is successful as follows

$$\theta^* = \sum_{g=1}^{N} m^g w^g \frac{c}{b^g}, \tag{21}$$

which is a weighted average of the cost-benefit ratio of different types of agents. Let $b_{min} = \min \{b^g\}_{g=1}^{N}$. The following proposition shows our previous results still hold when there is unobservable heterogeneity among agents.

Proposition 8 *Given a subsidy-tax program with $s < c$ and $s < t < b_{min}$, there exists a unique equilibrium in which a type j agent follows the strategy below.*

$$
\begin{aligned}
a = 1, \textit{Reject}, &\quad \textit{if} \;\; x \geq \eta_g^*(s,t), \\
a = 1, \textit{Accept}, &\quad \textit{if} \;\; \xi_g^*(s,t) \leq x < \eta_g^*(s,t), \\
a = 0, &\quad \textit{if} \;\; x < \xi_g^*(s,t),
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_g^*(s,t) &= \sum_{g=1}^{N} m^g w^g \frac{c-s}{b^g - t} + \sigma F_g^{-1}\left(\frac{c-s}{b_g - t}\right), \\
\eta_g^*(s,t) &= \sum_{g=1}^{N} m^g w^g \frac{c-s}{b^g - t} + \sigma F_g^{-1}\left(\frac{s}{t}\right).
\end{aligned}
$$

When $s = c$ and $c < t < b_{min}$, the equilibrium outcome converges to the first best outcome and the expected cost of the program converges to 0 when $\sigma \to 0$.

If agents also differ in the cost of investment, i.e. c^g can be different across groups, we need to relax the assumption that type are unobservable to the government. Instead, we assume the government can observe c_i for each individual agent. If $c_i = c^{g1} = c^{g2}$, the government doesn't need to know whether agent i is from $g1$ or $g2$. Under this setup, it is equivalent to solve the problem with $\tilde{c^g} = 1$ and $\tilde{b^g} = \frac{b^g}{c^g}$, and size up agent i's offer by c_i^g.

The intuition for how our proposed stimulus program works in the case with ex-ante heterogeneous agents is essentially the same as in the benchmark model. The stimulus program incentivizes "pivotal" agents who originally choose to walk away to change their decisions. All agents knowing that there is an increase in the aggregate action l all believe in a higher probability of success. Amplified by higher-order beliefs, the stimulus program can efficiently restore the first-best coordination results. Note that the notion of "pivotal" agents refers to the interim type of agents. Since different groups have different profitabilities from successful investments, they require different success probability to agree to invest. Our stimulus program identifies and targets at agents with beliefs right below the cutoffs of their own group. In Sakovics and Steiner (2012), they only look at direct subsidy programs and argue that an efficient program should target at the ex-ante "pivotal" group, in our setup, the group with low b^g and high w^g. Our results above demonstrate that by allowing an additional intervention tool, the contingent tax t, we are able to reduce coordination failure at a much lower cost. Moreover, the implementation of our proposed program doesn't require information on an agent's group, therefore our proposed program could save the potential cost of information acquisition.

4.2 General Payoff Structure

To address the potential concerns about the generality of the payoff structure, in this section, we follow the set-ups of the symmetric binary-action global game in Morris and Shin (2003) and allow for continuous payoff structure.

As in the benchmark model in section 2, an agent's payoff from $a_i = 0$ is normalized to zero. An agent's payoff from $a_i = 1$ is modified to be a continuous function $\pi(\theta, l)$, which increases in both the fundamental θ and the aggregate action $l = \int_0^1 a_i di$.

The stimulus program (s, t) consists of two parts, a direct subsidy $s \geq 0$ and a proportional tax $t \in [0, 1]$. If an agent accepts the offer, she receives a direct upfront subsidy s and pays the proportional tax after the realization of the investment outcome. Her payoff from accepting the offer is

$$u^i(a_i = 1, \text{Accept}) = (1 - t)\pi(\theta, l) + s. \tag{22}$$

Agents who receive low private signals believe in low realization of the fundamental θ and low aggregate action l are pessimistic about their payoffs from investments. Therefore, they expect to pay low tax and are more willing to accept the stimulus offer than optimistic agents. Recall the partial-participation programs in the benchmark model. These programs do not appeal to the optimistic agents who don't need extra incentive to invest, which efficiently saves resources and reduces the cost of the program. The proportional tax t captures this feature and helps to target at agents receiving medium signals.

We adopt the standard assumptions on the payoff function and the posterior distribution in the literature.

Assumption 3 *The payoff function $\pi(\theta, l)$ and the posterior distribution $g(\theta|x)$ satisfy the following properties for σ small enough:*

1. *(Strategic Complementarity) The payoff function $\pi(\theta, l)$ is weakly increasing in both arguments.*

2. *(Laplacian Action) $\int_0^1 \pi(\theta, l)\mathrm{d}l$ is continuous and strictly increasing in θ.*

3. *(Limit Dominance) There exists $\underline{\theta}, \bar{\theta}$ such that*

$$\pi(\theta, 1) < 0, \text{ for all } \theta < \underline{\theta}, \tag{23}$$

$$\pi(\theta, 0) > 0, \text{ for all } \theta > \bar{\theta}, \tag{24}$$

4. (Continuity) $\int_{-\infty}^{\infty} g(\theta|x)\pi(\theta, l)\mathrm{d}\theta$ *is continuous in* σ *and* x *for any* x.

The first assumption states the strategic complementarity among the action choices of different agents. The individual payoff of taking one action increases when more agents are taking the same action. Also, a higher fundamental increases everyone's incentive to choose $a_i = 1$, given the same aggregate action. Note that the payoff function need not be strictly increasing or continuous. For example, the payoff function in our simple model in section 2 is a step function. The role of the second assumption is to make sure uniqueness of the equilibrium cutoff. The third assumption is to ensure the existence of two dominance regions so that we can adopt the iterated deletion of dominated strategies from both sides. The last assumption regulates the limiting behavior of the equilibrium when the size of the information friction σ goes to 0.

The equilibrium without intervention is characterized in the proposition below. The "natural outcome" serves as a benchmark to analyze the effect of stimulus programs.

Proposition 9 *Without intervention* ($s = t = 0$)*, when the information friction* σ *is small enough, there is a unique equilibrium, in which each agents invests if and only if her private signal* $x \geq \xi$ *given by*

$$\int_0^1 \pi(\xi - \sigma F^{-1}(1-l), l)dl = 0.$$

In the limit when the information friction vanishes, $\lim_{\sigma \to 0} \xi = \xi_0^*$, where ξ_0^* is the unique solution to

$$\int_0^1 \pi(\xi_0^*, l)dl = 0. \tag{25}$$

Compare the coordination results characterized in the above proposition with the first best outcome. In the first-best scenario, if all agents investing can generate positive surplus, the social optimal outcome is for all agents to invest. In other words, the first best scenario is that all agents follow the same cutoff strategy $\underline{\theta}$, the upper bound for the left dominance region. The natural coordination outcome $\xi^* > \underline{\theta}$. Therefore if the realized fundamental $\theta \in (\xi^*, \underline{\theta})$, there would be a coordination failure. And the goal of intervention is to reduce the coordination threshold from ξ^* to as close to $\underline{\theta}$ as possible.

Next we analyze the equilibrium with stimulus package (s, t). We focus on the partial-participation programs and demonstrates its zero cost of implementation in the limiting case. Proposition 10 summarizes the conditions for such partial-participation programs.

Definition 1 *A stimulus program* (s, t) *is a partial-participation program with target* ξ^* *if and only if it satisfy the following three conditions:*

1. $\pi(-\infty, 1) < -\frac{s}{1-t}$,

2. $\pi(\xi^*, 1) \geq \frac{s}{t}$,

3. $\int_0^1 \pi(\xi^*, l)dl = -\frac{s}{1-t}$.

Proposition 10 *Given a partial-participation program* (s, t) *with target* ξ^**, when the information friction* σ *is small enough, there exists a unique Bayesian Nash equilibrium, in which all agents follow the same threshold strategy as follows,*

$$a_i = 1, \textit{Reject}, \quad \textit{if} \quad x_i \geq \eta^*(s,t), \tag{26}$$

$$a_i = 1, \textit{Accept}, \quad \textit{if} \quad \xi^*(s,t) \leq x_i < \eta^*(s,t), \tag{27}$$

$$a_i = 0, \quad \textit{if} \quad x_i < \xi^*(s,t), \tag{28}$$

where $\xi^*(s,t)$ *and* $\eta^*(s,t)$ *converge to* ξ^* *when* $\sigma \to 0$.

The above proposition provides conditions under which there exists partial-participation program to reduce the coordination threshold to ξ^* in the limit with zero noise. Same as in the benchmark model, in the limit, the middle signal region shrinks to zero, which implies a zero cost of implementation. The question left is whether there exist such program to costlessly restore the first best scenario, i.e. $\xi^* = \underline{\theta}$. The proposition below answers the question.

Proposition 11 *If the condition below is satisfied, for any* $\underline{\theta} < \xi < \xi_0^*$*, there exists a partial-participation program with target* ξ.

$$\int_0^1 \pi(\underline{\theta}, l)dl \geq \pi(-\infty, 1). \tag{29}$$

With the stimulus offer, all agents' perspectives of the coordination results become more optimistic. Therefore, the left dominance region, where agents prefer to choose $a = 0$ even if $l = 1$, shrinks. The condition specified in Proposition 11 is to make sure that the left dominance region still exists with the stimulus offer. If the condition is violated, we cannot prove the uniqueness of equilibrium. However, if we follow the equilibrium refinements proposed in Goldstein and Pauzner (2005), we can select the equilibrium described in Proposition 10 even without the left dominance region. Therefore, following the refinements, there always exists a partial-participation program to restore the best-best scenario. Moreover, the left dominance region may disappear because linear tax schedule t can be too generous to pessimistic agents. If the policy maker adds an extra tax in the case of disaster (when $\theta < \underline{\theta}$)

or adds convexity to the tax schedule properly, the left dominance region as well as the uniqueness of the equilibrium can be recovered. Either way, there always exists a stimulus program that can restore the first best scenario.

4.3 Finite Number of Players

In this section, we consider an alternative setup with N agents instead of a continuum of agents. The payoff from investment is modified accordingly to

$$\tilde{\pi}_N(\theta, n) = \pi(\theta, \frac{n}{N}), \tag{30}$$

where n is the number of agents choosing $a = 1$ and $\pi(\theta, l)$ is the payoff function in section 3.3.

Suppose other players follow a cutoff action strategy with threshold k, the expected payoff of choosing $a_j = 1$ when receiving a private signal x is

$$U_N(k, x) = \int_{-\infty}^{\infty} g(\theta|x) \sum_{j=1}^{N} \binom{N-1}{j-1} F\left(\frac{k-\theta}{\sigma}\right)^{N-j} \left(1 - F\left(\frac{k-\theta}{\sigma}\right)\right)^{j-1} \pi\left(\theta, \frac{j}{N}\right) \mathrm{d}\theta. \tag{31}$$

With finite number of agents, we can no longer apply the law of large numbers. Instead, given the realization of θ, the aggregate action $\frac{n}{N}$ follows a binomial distribution. However, $U_N(k, x)$ is still decreasing in k and increasing x, so we can apply iterated deletion of dominated strategies to pin down the unique equilibrium.

Following the analyses in proposition 10, everything goes through. The equilibrium action cutoff $\xi_N^*(s, t)$ and offer cutoff $\eta_N^*(s, t)$ are given by

$$U_N(\xi_N^*(s, t), \xi_N^*(s, t)) = -\tfrac{s}{1-t} \tag{32}$$

$$U_N(\xi_N^*(s, t), \eta_N^*(s, t)) = \tfrac{s}{t}. \tag{33}$$

Moreover,

$$\lim_{\sigma \to 0} \eta_N^*(s, t) = \lim_{\sigma \to 0} \xi_N^*(s, t). \tag{34}$$

Therefore, in the limit of zero noise, the probability distribution of aggregate action $\frac{n}{N}$ converges to a unit mass at value zero. And the main result of costless restoration of first best outcome still holds. Recall $\xi^*(s, t)$ and $\eta^*(s, t)$ are the action and offer cutoff in the model with a continuum of players. The following proposition establishes the relation between the game with finite number of agents and the game with a continuum of agents.

Proposition 12 *Given assumptions A1-A4, the investment and participation cutoffs in the finite player game converge to those in the game with a continuum of players.*

$$\lim_{N\to\infty} \xi_N^*(s,t) = \xi^*(s,t), \tag{35}$$

$$\lim_{N\to\infty} \eta_N^*(s,t) = \eta^*(s,t). \tag{36}$$

Proposition 12 above confirms that all results in the game with a continuum of players still hold for finite number of agents.

5 Selected Applications

The partial-participation programs can be applied to various contexts with coordination problems. In this section, we discuss three representative applications.

5.1 Debt Rollover

It has been widely recognized in the literature that panic-based debt run can leads to inefficient firm default. Specifically, consider a firm with many small debt-holders. The firm is more likely to survive if more debt-holders roll over their debts. Therefore, debt-holders' rollover decisions features strategic complementarities. When the fundamental of the firm is weak, debt-holders might stop rolling over their debts because they worry the others would also stop, which can leads to self-fulfilling debt run. Our analyses suggests tranching can be a cost-effective way to reduce such coordination failure. Instead of one standard debt contract, the firm can issue two types of debts with different seniorities. The senior debt promises lower return yet provides higher payment than the junior debt when the firm defaults. Without tranching, debt-holders, who have medium beliefs and coordination concerns, would not roll over their standard debts. With the safer option of senior debt, they are willing to lend to the firm which eases the liquidity concern of the firm and boosts all debt-holders' beliefs in the firm's survival. This effect can be amplified by higher order beliefs. In equilibrium, only the pivotal debt-holders choose the senior option. However, the availability of the safer senior debt improves all debt-holders' belief in that the firm can raise enough fund to survive.

Bank run is another similar application. To implement the partial-participation programs, the government can offer optional but costly deposit insurance. The insurance is costly if the bank survives yet provides protection when the bank fails. Therefore, it would work in a similar way as the senior debt option to reduce coordination failure. It is less costly than the mandatory deposit insurance because it screens out the "pivotal depositors" and leaves out the optimistic depositors who would not run even without insurance protection.

5.2 Market Freeze

During the 2008 financial crisis, many financial institutions and investors significantly reduced their leverage. This process pushed down the market prices of Commercial Mortgage-Backed Securities (CMBS) and Residual Mortgage-Backed Securities (RMBS). The markets for RMBS and CMBS froze, and prices were well below their fundamentals. Among others, coordination failure can prevent the market from thawing. If only a few investors participate in the market for Mortgage-Backed Securities (MBS), the liquidity in the market is not enough to drive the prices back to the fundamental and the participating investors suffer losses on their investments. However, if a significant amount of liquidity is injected in the market, the prices are more likely to be driven back to reflect the fundamental and investors who bought at a discount can profit from the investment.

In March of 2009, US Treasury announced the Legacy Securities Public-Private Investment Program (PPIP). Under the program, private equity was matched by government equity and debt to form Public-Private Investment Funds (PPIFs) and purchase highly rated legacy MBS from financial institutions. Private investors in the PPIFs effectively receive investment subsidies from the government and are levered up for their investment. They earn higher investment return in the good times and are protected by limited liabilities in the bad times. Hence, PPIP is uniformly beneficial to all qualifying private investors and can be interpreted as full-participation programs in our model. PPIP is not efficient in resource allocation in the sense that part of the government funding is provided to the optimistic investors who would have invest in MBS market without PPIP. According to our analyses, the government can reduce the cost of rejuvenating the market by offering a partial-participation program instead. Mapping into the context of PPIP, the government could offer to inject equity into PPIFs in proportion to debt holdings by private investors. This option of debt investment reduces the losses from freezing MBS market. As a return, the government shares the profit of investment if the market for MBS is successfully rejuvenated. This offer incentivizes the pivotal investors to invest in MBS market. Since all investors are aware of the offer, they know that the aggregate investment will increase and hence also have more incentive to invest.

5.3 Shopping Mall Investment

We analyze a real investment problem in this section. Consider a newly opened shopping mall inviting different brands to open new stores. Since all stores benefit from customers' visit to the shopping mall, all stores' investment return increases in the occupancy ratio in the shopping mall. Therefore, coordination failure could leads to low occupancy ratio and failure of the shopping mall. In order to boost investment, according to our analyses, the shopping mall manager could offer an equity injection option. Specifically, if a brand accepts the equity injection offer and opens a new store in the shopping mall, the shopping mall manager pays part of the investment cost and receives proportional profit made by the store as a return. This offer is not appealing to the optimistic brands because they don't want to share the profits with the shopping mall. For brands who are around investment threshold, the equity injection offer reduces their investment risk and increases their expected payoff from the investment. Amplified by higher-order beliefs, all brands significantly lower their investment threshold. Moreover, in equilibrium, only the "pivotal" brands accept the offer. Therefore the resources to finance the stimulus offer are effectively allocated.

It is reasonable to assume different brands have different profit functions. We have shown in section 3.2 that the interim critical agents who are around their own investment thresholds self-select to accept our offer. The result that the equity injection offer effectively reduces coordination failure and incurs low financing cost for the shopping mall owner still holds.

6 Conclusions

In this paper, we analyze a representative model with strategic complementarities in which coordination failure can lead to welfare loss and create room for interventions. We adopt global games techniques to pin down the unique threshold equilibrium and propose a stimulus program for a policy maker to lower the threshold and reduce coordination failures. The stimulus program is offered to agents who take the efficient action and allow them to receive upfront subsidies and pay a tax proportional to the agents' realized payoffs. Since the program charges a tax in the case of successful coordination, optimistic agents who believe in a high probability of success don't accept the offer but would take the efficient action anyway. In fact, only a small mass of agents who receive signals around the threshold self-select to accept the offer and be incentivized to take the efficient action. With more agents incentivized to take the efficient action, the expected payoff from taking the efficient action increases for all agents. Therefore, agents receiving even lower signals are willing to take the efficient action. Repeat the thought process. It can be shown that the effect of the stimulus program is amplified by higher-order beliefs, and the threshold is significantly lowered. In the limit of zero noise in agents' private signals, our proposed program eliminates all coordination failures at zero cost since the mass of "pivotal agents" goes to zero.

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Appendices

A Proofs

Proof of Proposition 1. It can be proved by iterated deletion of dominated strategies. Without the stimulus program, agents choose to invest if and only if $p(x;k) \geq \frac{b}{c}$. First, we want to show that strategies survive n rounds of iterated deletion of dominated strategies if and only if

$$a(x) = 0, \text{ if } x < \underline{\xi}_n, \tag{A.1}$$
$$\text{and } a(x) = 1, \text{ if } x \geq \bar{\xi}_n. \tag{A.2}$$

where $\left\{(\underline{\xi}_n, \bar{\xi}_n)\right\}_{n=0}^{\infty}$ satisfies

$$-\infty = \underline{\xi}_0 < \underline{\xi}_1 \leq \cdots \leq \underline{\xi}_n \leq \cdots \leq \bar{\xi}_n \leq \cdots \leq \bar{\xi}_1 < \bar{\xi}_0 = +\infty. \tag{A.3}$$

This result can be proved by induction. Let $\underline{\xi}_0 = -\infty$ and $\bar{\xi}_0 = +\infty$, so the first round of deletion starts with the full set of strategies. Suppose round $n \in \mathbb{N}$ of deletion has been completed. In round $n+1$, the best scenario for an agent to invest is that all other agents follow a cutoff strategy with threshold $\underline{\xi}_n$. Therefore, for any x such that $p(x;\underline{\xi}_n) < \frac{c}{b}$, $a(x) = 1$ is strictly worse than $a(x) = 0$. Similarly, the best scenario for an agent to choose $a_i = 1$ is that all other agents follow a cutoff strategy with threshold $\underline{\xi}_n$. As a result, for x such that $p(x;\bar{\xi}_n) > 0$, any strategy profile with $a(x) = 1$ is strictly worse than $a(x) = 0$.

Given $p(x;k)$ is non-decreasing in x, the strategy profiles that survives deletion of dominated strategies can be summarized in the form of (A.1)(A.2), with $(\underline{\xi}_{n+1}, \bar{\xi}_{n+1})$ defined inductively as

$$\underline{\xi}_{n+1} = \inf\left\{x : p(x;\underline{\xi}_n) \geq \frac{c}{b}\right\} \tag{A.4}$$

and

$$\bar{\xi}_{n+1} = \sup\left\{x : p(x;\bar{\xi}_n) \leq \frac{c}{b}\right\} \tag{A.5}$$

The monotonicity of $p(x;k)$ guarantees that $\underline{\xi}_{n+1} \leq \bar{\xi}_{n+1}$ given $\underline{\xi}_n \leq \bar{\xi}_n$. Note the dominance region assumption implies that $\underline{\xi}_1 > -\infty$ and $\bar{\xi}_1 < +\infty$ when σ is small enough. Therefore, $\left\{(\underline{\xi}_n, \bar{\xi}_n)\right\}_{n=0}^{\infty}$ is a well-defined sequence of real couple which satisfies (A.3).

Now we've proved that $\{\underline{\xi}_n\}_{n=1}^{\infty}$ and $\{\bar{\xi}_n\}_{n=1}^{\infty}$ are both monotonic and bounded sequences. Thus, they converges to two finite numbers $\underline{\xi}$ and $\bar{\xi}$ respectively when $n \to \infty$. And the two limits satisfy

$$\underline{\xi} \leq \bar{\xi}. \tag{A.6}$$

The definition (A.4)(A.5) implies that $p(\underline{\xi};\underline{\xi}) \geq \frac{c}{b}$ and $p(\bar{\xi};\bar{\xi}) \leq \frac{c}{b}$. Note that

$$p(\xi;\xi) = F\left(\frac{\xi - \theta^*(\xi)}{\sigma}\right) = \theta^*(\xi), \tag{A.7}$$

is strictly increasing in ξ. Therefore $\underline{\xi} = \bar{\xi}$ must be the unique solution to $\theta^*(\xi) = \frac{c}{b}$, which is

$$\xi_0^* = \frac{c}{b} + \sigma F^{-1}\left(\frac{c}{b}\right). \tag{A.8}$$

Since there's only one strategy that survives the iterated deletion of dominated strategies, the equilibrium of the game is unique and the associated equilibrium strategy is the cutoff investment strategy with threshold ξ_0^*. ■

Lemma 1 *Suppose the optimal strategy of an agent as a function of her posterior belief of success $\hat{p}_i$ can be characterized as*

$$\begin{aligned} a_i = 1, \textit{Reject}, &\quad \textit{if} \;\; \hat{p}_i > p_2^*, \\ a_i = 1, \textit{Accept}, &\quad \textit{if} \;\; p_1^* < \hat{p}_i \leq p_2^*, \\ a_i = 0, &\quad \textit{if} \;\; \hat{p}_i \leq p_1^*, \end{aligned}$$

where p_1^* and p_2^* are two threshold beliefs that satisfy $0 \leq p_1^* < p_2^* \leq 1$. There is a unique Bayesian Nash equilibrium and the equilibrium strategy of any agent is

$$\begin{aligned} a_i = 1, \textit{Reject}, &\quad \textit{if} \;\; x_i \geq \eta^*, \\ a_i = 1, \textit{Accept}, &\quad \textit{if} \;\; \xi^* \leq x_i < \eta^*, \\ a_i = 0, &\quad \textit{if} \;\; x_i < \xi^*, \end{aligned}$$

where $\xi^* = p_1^* + \sigma F^{-1}(p_1^*)$ and $\eta^* = p_1^* + \sigma F^{-1}(p_2^*)$

Proof of Lemma 1. We want to find a sequence $\left\{(\underline{\xi}_n, \bar{\xi}_n)\right\}_{n=0}^{\infty}$ such that strategies survives

n rounds of iterated deletion of dominated strategies only if

$$a(x) = 0, \text{ if } x < \underline{\xi}_n, \tag{A.9}$$

$$\text{and } a(x) = 1, \text{ if } x \geq \bar{\xi}_n. \tag{A.10}$$

The reason that we can only iterate on the investment cutoff without keeping track of the participation decisions is that an agent's investment decision is independent of other agents' participation decisions. The recursive expression for $\left\{(\underline{\xi}_n, \bar{\xi}_n)\right\}_{n=0}^{\infty}$ is

$$\underline{\xi}_{n+1} = \inf\{x : p(x; \underline{\xi}_n) \geq p_1^*\}, \tag{A.11}$$

$$\bar{\xi}_{n+1} = \sup\{x : p(x; \bar{\xi}_n) \leq p_1^*\}. \tag{A.12}$$

Applying the same techniques in the proof of Proposition 1, it becomes clear that the limit of the two cutoff sequences converges to

$$\xi^*(s,t) = p_1^* + \sigma F^{-1}\left(p_1^*\right), \tag{A.13}$$

which is the investment cutoff in the unique Bayesian Nash equilibrium of the global game. The associated participation cutoff η is the solution to

$$p(\eta; \xi^*(s,t)) = p_2^*. \tag{A.14}$$

Solving the above equation yields

$$\eta^*(s,t) = p_1^* + \sigma F^{-1}\left(p_2^*\right). \tag{A.15}$$

■

Proof of Proposition 2.

In case 1, *invest-and-reject* is dominated by *invest-and-accept*. Therefore, we can rewrite the investment payoff by letting $b' = b - t$ and $c' = c - s$ and directly apply Proposition 1. Similarly, *invest-and-accept* is jointly dominated by *invest-and-accept* and *not-invest* in case 3. Since the stimulus program is never going to be accepted, the equilibrium is the same as that described in Proposition 1. Case 2 is a direct implication of Lemma 1. ■

Proof of Proposition 3. Let $f(\xi) = \frac{c-b\xi}{1-\xi}$. Note that $\xi \in [0, \frac{c}{b})$, therefore $f(\xi) \in (0, c]$. Firstly, we find the full-participation programs. If (s,t) satisfies the following two conditions, it is a valid the full-participation program targeting at cutoff ξ.

1. $0 \le t \le f(\xi)$
2. $s = c - (b - t)\xi$

Note that the second condition makes sure the target equals ξ. Next step is to check the full-participation condition.

$$\frac{s}{t} = \frac{c - (b - t)\xi}{t} = \frac{c - b\xi}{t} + \xi$$

The ratio decreases strictly in t, therefore $\frac{s}{t} \ge \frac{c-b\xi}{f(\xi)} + \xi = 1$.

Next, we search for the partial-participation programs. If (s', t') satisfies the following two conditions, it is a valid full-participation program targeting at cutoff ξ.

1. $f(\xi) < t' \le b$
2. $s' = c - (b - t)\xi$

Note that the second condition makes sure the target equals ξ. Next check the partial-participation condition.

$$\frac{s'}{t'} = \frac{c - (b - t')\xi}{t'} = \frac{c - b\xi}{t'} + \xi$$

The ratio decreases strictly in t, therefore $\frac{s'}{t'} \in (\frac{c-b\xi}{b} + \xi, \frac{c-b\xi}{f(\xi)} + \xi] = [\frac{c}{b}, 1)$. It can be shown that the above conditions are also necessary.

Lastly, we calculate the cost function in the limit.

$$\lim_{\sigma \to 0} C(\theta, s, t) = \lim_{\sigma \to 0} (\tau s - t) \left[1 - F\left(\frac{\xi - \theta}{\sigma}\right)\right]$$

$$\lim_{\sigma \to 0} C(\theta, s', t') = \lim_{\sigma \to 0} (\tau s' - t') \left[F\left(\frac{\xi - \theta}{\sigma}\right) - F\left(\frac{\xi - \theta}{\sigma}\right)\right]$$

Therefore, $\forall \theta > \xi$ $\lim_{\sigma \to 0} C(\theta, s, t) = \lim_{\sigma \to 0} (\tau s - t) [1 - F(-\infty)] = \tau s - t$; and $\forall \theta < \xi$ $\lim_{\sigma \to 0} C(\theta, s, t) = \lim_{\sigma \to 0} (\tau s - t) [1 - F(\infty)] = 0$. And $\forall \theta \neq \xi$, $\lim_{\sigma \to 0} C(\theta, s', t') = 0$. ■

Proof of Proposition 4. We compare the expected cost of a full-participation program with (s, t) a partial-participation program (s', t') with small enough $\lambda > 0$.

The expected cost of the full-participation program is

$$\mathbb{E}_\theta[C(\theta, s, t)] = \frac{\tau s}{\bar{\theta} - \underline{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \left[1 - F\left(\frac{\xi^* - \theta}{\sigma}\right)\right] \mathrm{d}\theta - \frac{t}{\bar{\theta} - \underline{\theta}} \int_{\theta^*}^{\bar{\theta}} \left[1 - F\left(\frac{\xi^* - \theta}{\sigma}\right)\right] \mathrm{d}\theta,$$

and that of the partial-participation program (s',t'),

$$\mathbb{E}_\theta[C(\theta,s',t')] = \frac{\tau s'}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\left[F\left(\frac{\eta^*-\theta}{\sigma}\right)-F\left(\frac{\xi^*-\theta}{\sigma}\right)\right]\mathrm{d}\theta - \frac{t'}{\bar{\theta}-\underline{\theta}}\int_{\theta^*}^{\bar{\theta}}\left[F\left(\frac{\eta^*-\theta}{\sigma}\right)-F\left(\frac{\xi^*-\theta}{\sigma}\right)\right]\mathrm{d}\theta,$$

where ξ^* and η^* are the investment threshold and participation threshold defined as in Proposition 2, $\xi^* = \theta^* + \sigma F^{-1}(\theta^*)$, $\eta^*(s',t') = \theta^* + \sigma F^{-1}\left(\frac{s'}{t'}\right)$. To suppress notations, we omit the dependence of η^* on (s',t'). The difference between the cost of full-participation program (s,t) and that of partial-participation program (s',t') can be decomposed into two parts, $\mathbb{E}_\theta[C(\theta,s,t)] - \mathbb{E}_\theta[C(\theta,s',t')] = \Delta_1 + \Delta_2$, where

$$\Delta_1 = \frac{\tau s - t}{\bar{\theta}-\underline{\theta}}\int_{\theta^*}^{\bar{\theta}}\left[1-F\left(\frac{\eta^*-\theta}{\sigma}\right)\right]\mathrm{d}\theta + \frac{\tau s}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\theta^*}\left[1-F\left(\frac{\eta^*-\theta}{\sigma}\right)\right]\mathrm{d}\theta,$$

$$\Delta_2 = -\frac{\tau\theta^*(t'-t)}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\left[F\left(\frac{\eta^*-\theta}{\sigma}\right)-F\left(\frac{\xi^*-\theta}{\sigma}\right)\right]\mathrm{d}\theta + \frac{t'-t}{\bar{\theta}-\underline{\theta}}\int_{\theta^*}^{\bar{\theta}}\left[F\left(\frac{\eta^*-\theta}{\sigma}\right)-F\left(\frac{\xi^*-\theta}{\sigma}\right)\right]\mathrm{d}\theta.$$

Δ_1 and Δ_2 are the cost difference on the extensive margin and intensive margin respectively.

Notice $\mathbb{E}[C(\theta,s,t)]$ is linear in s and t. Therefore, to show the proposed partial-participation program $(s',t') = (\frac{c-\theta^*b}{1-\theta^*}+\theta^*\lambda, \frac{c-\theta^*b}{1-\theta^*}+\lambda)$ with small positive λ has lower cost than any full-participation program, we only need to consider two full-participation programs on the boundary, $\lambda_1 = 0$ with $(s,t) = (\frac{c-\theta^*b}{1-\theta^*}, \frac{c-\theta^*b}{1-\theta^*})$, and $\lambda_2 = -\frac{c-\theta^*b}{1-\theta^*}$, with $(s,t) = (c-\theta^*b, 0)$.

Consider the first case $(s,t) = (c-\theta^*b, 0)$. Plugging into the expression for Δ_1, we have

$$\begin{aligned}
\Delta_1 =& \frac{\tau(c-\theta^*b)}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\left[1-F\left(\frac{\eta^*-\theta}{\sigma}\right)\right]\mathrm{d}\theta,\\
=& \frac{\tau(c-\theta^*b)}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\int_{\eta^*}^{\bar{\theta}+\frac{1}{2}\sigma}\frac{1}{\sigma}f\left(\frac{x-\theta}{\sigma}\right)\mathrm{d}x\mathrm{d}\theta,\\
=& \frac{\tau(c-\theta^*b)}{\bar{\theta}-\underline{\theta}}\int_{\eta^*}^{\bar{\theta}+\frac{1}{2}\sigma}\int_{\underline{\theta}}^{\bar{\theta}}\frac{1}{\sigma}f\left(\frac{x-\theta}{\sigma}\right)\mathrm{d}\theta\mathrm{d}x,\\
=& \frac{\tau(c-\theta^*b)}{\bar{\theta}-\underline{\theta}}\int_{\eta^*}^{\bar{\theta}+\frac{1}{2}\sigma}\left[1-F\left(\frac{x-\bar{\theta}}{\sigma}\right)\right]\mathrm{d}x,\\
=& \frac{\tau(c-\theta^*b)}{\bar{\theta}-\underline{\theta}}\left[\bar{\theta}+\frac{1}{2}\sigma-\eta^*-\sigma\int_{-\frac{1}{2}}^{\frac{1}{2}}F(y)\mathrm{d}y\right] > \frac{\tau(c-\theta^*b)}{\bar{\theta}-\underline{\theta}}(1-\theta^*),
\end{aligned}$$

which is strictly positive.

For Δ_2, notice

$$\begin{aligned}\int_\alpha^\beta \left[F\left(\frac{\eta^*-\theta}{\sigma}\right) - F\left(\frac{\xi^*-\theta}{\sigma}\right)\right] \mathrm{d}\theta &= \int_\alpha^\beta \int_{\xi^*}^{\eta^*} \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) \mathrm{d}x\mathrm{d}\theta, \\ &= \int_{\xi^*}^{\eta^*} \int_\alpha^\beta \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) \mathrm{d}\theta\mathrm{d}x, \\ &= \int_{\xi^*}^{\eta^*} \left[F\left(\frac{x-\alpha}{\sigma}\right) - F\left(\frac{x-\beta}{\sigma}\right)\right] \mathrm{d}x,\end{aligned}$$

therefore

$$\begin{aligned}\Delta_2 &= \left(\frac{c-\theta^* b}{1-\theta^*}+\varepsilon\right)\frac{1}{\bar{\theta}-\underline{\theta}}\left[\int_{\xi^*}^{\eta^*} F\left(\frac{x-\theta^*}{\sigma}\right)\mathrm{d}x - \tau\theta^*(\eta^*-\xi^*)\right], \\ &= \left(\frac{c-\theta^* b}{1-\theta^*}+\varepsilon\right)\frac{\theta^*(\eta^*-\xi^*)}{\bar{\theta}-\underline{\theta}}\left[\int_{F^{-1}(\theta^*)}^{F^{-1}\left(\frac{s'}{t'}\right)} \frac{F(y)}{\theta^*(F^{-1}\left(\frac{s'}{t'}\right)-F^{-1}(\theta^*))}\mathrm{d}y - \tau\right], \\ &= \left(\frac{c-\theta^* b}{1-\theta^*}+\varepsilon\right)\frac{\theta^*(\eta^*-\xi^*)}{\bar{\theta}-\underline{\theta}}\left[G\left(\theta^*,\frac{s'}{t'}\right)-\tau\right].\end{aligned}$$

Taking λ to 0, we have

$$\lim_{\lambda\to 0^+}\Delta_2 = \left(\frac{c-\theta^* b}{1-\theta^*}\right)\frac{\theta^*\sigma(\frac{1}{2}-F^{-1}(\theta^*))}{\bar{\theta}-\underline{\theta}}[G(\theta^*,1)-\tau] = \frac{c-\theta^* b}{\bar{\theta}-\underline{\theta}}\theta^*\sigma[G(\theta^*,1)-\tau].$$

If the first condition holds, $\tau < G(\theta^*,1)$, $\lim_{\varepsilon\to 0^+}\Delta_2 > 0$, $\Delta_1+\Delta_2$ is strictly positive for small enough λ. Also, if the second condition holds, $\theta^*+\sigma<1$,

$$\lim_{\lambda\to 0^+}\Delta_1+\Delta_2 > \frac{\tau(c-\theta^* b)}{\bar{\theta}-\underline{\theta}}(1-\theta^*-\theta^*\sigma) > 0.$$

Now let's turn to the second case with $s=t=\frac{c-\theta^* b}{1-\theta^*}$. For Δ_1, since $\eta^*=\theta^*+\sigma F^{-1}(\frac{s'}{t'}) < \theta^*+\frac{1}{2}\sigma$, we have

$$\Delta_1 > \frac{(\tau-1)s}{\bar{\theta}-\underline{\theta}}\int_{\theta^*}^{\bar{\theta}}\left[1-F\left(\frac{\theta^*+\frac{1}{2}\sigma-\theta}{\sigma}\right)\right]\mathrm{d}\theta + \frac{\tau s}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\theta^*}\left[1-F\left(\frac{\theta^*+\frac{1}{2}\sigma-\theta}{\sigma}\right)\right]\mathrm{d}\theta \geq 0.$$

The last inequality is strict when $\tau>1$. For Δ_2, we have

$$\Delta_2 = \lambda\frac{\sigma\theta^*(F^{-1}\left(\frac{s'}{t'}\right)-F^{-1}(\theta^*))}{\bar{\theta}-\underline{\theta}}\left[G\left(\theta^*,\frac{s'}{t'}\right)-\tau\right].$$

If $\tau > 1$, $\lim_{\lambda\to 0^+}\Delta_1 > 0$, $\lim_{\lambda\to 0^+}\Delta_2 = 0$. Thus, $\mathbb{E}_\theta[C(\theta,s,t)] - \mathbb{E}_\theta[C(\theta,s',t')] =$

$\Delta_1 + \Delta_2 > 0$ for small enough λ.

If $\tau = 1$, since $\frac{s'}{t'} > \frac{c}{b} > \theta^*$, $G\left(\theta^*, \frac{s'}{t'}\right) > 1 = \tau$, $\Delta_2 > 0$ for any positive λ. Combining with $\Delta_1 \geq 0$, we have $\mathbb{E}_\theta[C(\theta, s, t)] - \mathbb{E}_\theta[C(\theta, s', t')] = \Delta_1 + \Delta_2 > 0$ for any positve λ.

To sum up, in either case, when λ being positve and small enough, the partial particiaption program $(s', t') = (\frac{c-\theta^* b}{1-\theta^*} + \theta^*\lambda, \frac{c-\theta^* b}{1-\theta^*} + \lambda)$ has lower expected cost than any full-participation program targeting at θ^*. ■

Proof of Proposition 6. If we can choose (s, t) properly such that $0 < p_1^*(s, t) < p_2^*(s, t) < 1$, Lemma 1 implies in the unique Bayesian Nash equilibrium, agents follow a threshold strategy

$$\begin{aligned} a_i = 1, \text{Reject}, &\quad \text{if} \quad x_i \geq \eta^*(s, t), \\ a_i = 1, \text{Accept}, &\quad \text{if} \quad \xi^*(s, t) \leq x_i < \eta^*(s, t), \\ a_i = 0, &\quad \text{if} \quad x_i < \xi^*(s, t), \end{aligned}$$

where

$$\begin{aligned} \xi^*(s, t) &= p_1^*(s, t) + \sigma F^{-1}(p_1^*(s, t)), \\ \eta^*(s, t) &= p_1^*(s, t) + \sigma F^{-1}(p_2^*(s, t)). \end{aligned}$$

Moreover, $\xi^*(s, t)$ and $\eta^*(s, t)$ both converges to $p_1^*(s, t)$ when $\sigma \to 0$. Thus, for any continuous belief of the fundamental held by the government, the ex-ante cost of the program converges to 0 when $\sigma \to 0$.

Now we want to show that it is possible to choose (s, t) such that $0 < p_1^*(s, t) < p_2^*(s, t) < 1$ and $p_1^*(s, t)$ can be arbitrarily close to 0. Let $s = c - \varepsilon$ and $\frac{c+c^e-\gamma b}{1-\gamma} < t < b$. The choice of t is feasible since Assumption 1b implies $\frac{c+c^e-\gamma b}{1-\gamma} < b$. Note $\frac{c+c^e-\gamma b}{1-\gamma} < t$ implies

$$\begin{aligned} p_2^*(s, t) &= \frac{s}{\gamma b + t(1-\gamma) - c^e} < \frac{c - \varepsilon}{c}, \\ p_1^*(s, t) &= \frac{c - s}{(1-\gamma)(b - t)} = \frac{\varepsilon}{(1-\gamma)(b-t)}. \end{aligned}$$

Therefore, for any fixed t, when $\varepsilon \to 0$, $p_1^*(s, t)$ converges to 0 and $p_2^*(s, t)$ converges to a positive number which is strictly less than 1. ■

Proof of Proposition 7. The proof is similar to the proof of Lemma 1. We want to find a sequence $\left\{(\underline{\xi}_n^g, \bar{\xi}_n^g)_{g=1}^N\right\}_{n=0}^{\infty}$ such that the strategies of group g agents survives n rounds of iterated deletion of dominated strategies only if

$$a^g(x) = 0, \text{ if } x < \underline{\xi}_n^g, \tag{A.16}$$

$$\text{and } a^g(x) = 1, \text{ if } x \geq \bar{\xi}_n^g. \tag{A.17}$$

To simplify notations, let $\underline{\xi}_n = (\underline{\xi}_n^g)_{g=1}^N$ and $\bar{\xi}_n = (\bar{\xi}_n^g)_{g=1}^N$ be the vectors of threshold signals. The recursive expression for $\left\{(\underline{\xi}_n^g, \bar{\xi}_n^g)_{g=1}^N\right\}_{n=0}^{\infty}$ is

$$\underline{\xi}_{n+1}^g = \inf_x \{x : p^g(x; \underline{\xi}_n) \geq \frac{c}{b^g}\}, \tag{A.18}$$

$$\bar{\xi}_{n+1}^g = \sup_x \{x : p^g(x; \bar{\xi}_n) \leq \frac{c}{b^g}\}. \tag{A.19}$$

We can prove by induction that

$$-\infty = \underline{\xi}_0 < \underline{\xi}_1 \leq \cdots \leq \underline{\xi}_n \leq \cdots \leq \bar{\xi}_n \leq \cdots \leq \bar{\xi}_1 < \bar{\xi}_0 = +\infty. \tag{A.20}$$

Since any bounded sequence has a finite limit, take n to ∞, we have

$$\bar{\xi} \geq \underline{\xi}. \tag{A.21}$$

Now we want to show $\bar{\xi} = \underline{\xi}$. It can be proved by contradiction. Suppose $\bar{\xi} > \underline{\xi}$. Let h be the group such that $\bar{\xi}^h - \underline{\xi}^h = \max_g \left\{\bar{\xi}^g - \underline{\xi}^g\right\} > 0$. Note that $\theta^*(\bar{\xi})$ is the solution to

$$\sum_{g=1}^{N} w^g m^g F^g \left(\frac{\bar{\xi}^g - \theta}{\sigma}\right) = \theta. \tag{A.22}$$

Therefore, $\theta^*(\bar{\xi}) - (\bar{\xi}^h - \underline{\xi}^h)$ is the solution to

$$\sum_{g=1}^{N} w^g m^g F^g \left(\frac{\bar{\xi}^g - (\bar{\xi}^h - \underline{\xi}^h) - \theta}{\sigma}\right) - \theta - (\bar{\xi}^h - \underline{\xi}^h) = 0. \tag{A.23}$$

Also notice $\theta^*(\underline{\xi})$ is the solution to

$$\sum_{g=1}^{N} w^g m^g F^g \left(\frac{\underline{\xi}^g - \theta}{\sigma}\right) - \theta = 0. \tag{A.24}$$

Let's (A.23) and (A.24). Since $\underline{\xi}^g > \bar{\xi}^g - (\bar{\xi}^h - \underline{\xi}^h)$ and $\bar{\xi}^h - \underline{\xi}^h > 0$, the left hand side of (A.24) is strictly larger than the left hand side of (A.24) for any given θ. Given the left hand side of A.24 is strictly decreasing in θ, we must have $\theta^*(\bar{\xi}) - (\bar{\xi}^h - \underline{\xi}^h) < \theta^*(\underline{\xi})$. Therefore,

$$\begin{aligned} p^h(\bar{\xi}^h;\bar{\xi}) =& Pr^h[\theta > \theta^*(\bar{\xi})|\bar{\xi}^h], \\ =& F^h\left(\frac{\bar{\xi}^h - \theta^*(\bar{\xi})}{\sigma}\right), \\ =& F^h\left(\frac{\underline{\xi}^h - [\theta^*(\bar{\xi}) - (\bar{\xi}^h - \underline{\xi}^h)]}{\sigma}\right), \\ >& F^h\left(\frac{\underline{\xi}^h - \theta^*(\theta^*(\underline{\xi})}{\sigma}\right), \\ =& p^h(\underline{\xi}^h;\bar{\xi}). \end{aligned}$$

However. (A.18) and (A.19) implies $p^h(\bar{\xi}^h;\bar{\xi}) = p^h(\underline{\xi}^h;\underline{\xi}) = \frac{c}{b^h}$. Contradiction. This implies $\bar{\xi} = \underline{\xi} = \xi_0$.

To solve for ξ_0, note ξ_0 and θ_0 are the solutions to

$$\sum_{g=1}^{N} w^g m^g F^g\left(\frac{\xi^g - \theta}{\sigma}\right) = \theta, \tag{A.25}$$

$$F^g\left(\frac{\xi^g - \theta}{\sigma}\right) = \frac{c}{b^g}, \quad \text{for any } g = 1, \ldots, N. \tag{A.26}$$

Plugging (A.26) into (A.25) we have

$$\theta_0 = \sum_{g=1}^{N} m^g w^g \frac{c}{b^g}, \tag{A.27}$$

$$\xi_0^g = \sum_{g=1}^{N} m^g w^g \frac{c}{b^g} + \sigma F_g^{-1}\left(\frac{c}{b^g}\right), \quad \text{for any } g = 1, \ldots, N. \tag{A.28}$$

■

Proof of Proposition 8. The optimal response of an agent in group g is

$$\begin{aligned} a_i = 1, \text{Reject}, &\text{ if } \hat{p}_i \geq \frac{s}{t}, \\ a_i = 1, \text{Accept}, &\text{ if } \frac{c-s}{b^g - t} \leq \hat{p}_i < \frac{s}{t}, \\ a_i = 0, &\text{ if } \hat{p}_i < \frac{c-s}{b^g - t}; \end{aligned}$$

We can apply the same method in the proof of Proposition 7 and show that in any equilibrium, agents of group g invest if and only if their private signal is greater or equal to

$$\xi_g^*(s,t) = \sum_{g=1}^{N} m^g w^g \frac{c-s}{b^g - t} + \sigma F_g^{-1}\left(\frac{c-s}{b_g - t}\right). \tag{A.29}$$

Given the investment thresholds, we know the fundamental threshold above which there will be successful investment is

$$\theta^*(s,t) = \sum_{g=1}^{N} m^g w^g \frac{c-s}{b^g - t}. \tag{A.30}$$

Therefore, the signal $\eta^*(s,t)$ that makes an agent from group g indifferent between accepting and rejecting the stimulus program is

$$\eta^*(s,t) = \sum_{g=1}^{N} m^g w^g \frac{c-s}{b^g - t} + \sigma F_g^{-1}\left(\frac{s}{t}\right). \tag{A.31}$$

■

Proof of Proposition 9. The posterior distribution of fundamental θ given signal x is

$$g(\theta|x) = \frac{1}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) \tag{A.32}$$

When $\sigma \to 0$, the posterior distribution of θ converges to a delta function centered at x.

Consider an agent who receives private signal x and knows that all other agents follow the action cutoff strategy k. The expected utility resulting from taking action 1 is

$$U(k,x) = \int_{-\infty}^{\infty} g(\theta|x)\pi\left(\theta, 1 - F\left(\frac{k-\theta}{\sigma}\right)\right) d\theta$$

Note that $U(k,x)$ weakly decreases in k and weakly increases in x. Intuitively, an agent benefit more from taking action $a=1$ if everyone else is more willing to choose $a=1$ or the agent receives a high signal indicating a high fundamental θ. Also note that the limiting behavior of the posterior distribution and the continuity of the expected utility in σ implies that expected utility inherits properties, such as the dominance region assumption from the payoff function when σ is small enough.

Next prove the uniqueness of equilibrium by iterated deletion of dominated strategies. The strategy profile of an agent is the action as a function of the private signal received. We denote it by $a(x) : \mathbb{R} \to \{0,1\}$. We will prove that strategy survives n rounds of iterated deletion of dominated strategies if and only if

$$a(x) = 0, \text{ if } x < \underline{\xi}_n, \tag{A.33}$$

$$\text{and } a(x) = 1, \text{ if } x \geq \bar{\xi}_n. \tag{A.34}$$

where $\left\{(\underline{\xi}_n, \bar{\xi}_n)\right\}_{n=0}^{\infty}$ satisfies

$$-\infty = \underline{\xi}_0 < \underline{\xi}_1 \leq \cdots \leq \underline{\xi}_n \leq \cdots \leq \bar{\xi}_n \leq \cdots \leq \bar{\xi}_1 < \bar{\xi}_0 = +\infty. \tag{A.35}$$

This result can be proved by induction. Let the starting node be $\underline{\xi}_0 = -\infty$ and $\bar{\xi}_0 = +\infty$, meaning that there is no restrictions on agents' strategy. Suppose round $n \in \mathbb{N}$ of deletion has been completed. In round $n+1$, the most optimistic belief for an agent is that all other agents follow a cutoff strategy with threshold $\underline{\xi}_n$. Therefore, for any x such that $U(\underline{\xi}_n, x) < 0$, $a(x) = 1$ is strictly dominated by $a(x) = 0$. Similarly, the most pessimistic belief for an agent is that all other agents follow a cutoff strategy with threshold $\bar{\xi}_n$. As a result, for x such that $U(\bar{\xi}_n, x) > 0$, any strategy profile with $a(x) = 0$ is strictly dominated by $a(x) = 1$.

Given $U(k, x)$ is non-decreasing in x, the strategy profiles that survives deletion of dominated strategies must satisfy the restrictions in (A.33) and (A.34), with $(\underline{\xi}_{n+1}, \bar{\xi}_{n+1})$ defined inductively as

$$\underline{\xi}_{n+1} = \inf\{x : U(\underline{\xi}_n, x) \geq 0\} \tag{A.36}$$

and

$$\bar{\xi}_{n+1} = \sup\{x : U(\bar{\xi}_n, x) \leq 0\} \tag{A.37}$$

The monotonicity of $U(k, x)$ guarantees that $\underline{\xi}_{n+1} \leq \bar{\xi}_{n+1}$. Note that the dominance region assumption implies that $\underline{\xi}_1 > -\infty$ and $\bar{\xi}_1 < +\infty$ when σ is small enough. Therefore, $\left\{(\underline{\xi}_n, \bar{\xi}_n)\right\}_{n=0}^{\infty}$ is a well-defined sequence of real couples which satisfies (A.35).

Now we've proved that $\{\underline{\xi}_n\}_{n=1}^{\infty}$ and $\{\bar{\xi}_n\}_{n=1}^{\infty}$ are both monotonic and bounded sequences. Thus, they converges to two finite numbers $\underline{\xi}$ and $\bar{\xi}$ respectively when $n \to \infty$. The definition (A.36) and (A.37) imply that $U(\underline{\xi}, \underline{\xi}) \geq 0$ and $U(\bar{\xi}, \bar{\xi}) \leq 0$.

$$U(\xi, \xi) = \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{\sigma} f\left(\frac{\xi - \theta}{\sigma}\right) \pi\left(\theta, 1 - F\left(\frac{\xi - \theta}{\sigma}\right)\right) d\theta = \int_0^1 \pi\left(\xi - \sigma F^{-1}(1 - l), l\right) dl$$

Since $U(\xi, \xi)$ strictly increases in ξ and $\underline{\xi} \leq \bar{\xi}$, it must be the case that $U(\underline{\xi}, \underline{\xi}) = U(\bar{\xi}, \bar{\xi}) = 0$.

By assumption, there is a unique solution to $U(\xi, \xi) = 0$. Denote the solution as ξ. Therefore, $\underline{\xi} = \bar{\xi} = \xi$. By the continuity of $U(k, x)$, we can calculate the limit of the unique threshold as the information friction vanishes as follows,

$$\lim_{\sigma \to 0} \xi = \xi^*,$$

where xi^* is the unique solution to

$$\lim_{\sigma \to 0} U(\xi^*, \xi^*) = \int_0^1 \pi(\xi^*, l) dl = 0.$$

■

Proof of Proposition 10. We first prove a lemma that will useful in the main proof of uniqueness.

Lemma 2 *Given that all other agents follow the same action cutoff strategy k: choose $a_i = 1$ if the private signal $x_i \geq k$ and $a_i = 0$ otherwise, the agent's optimal investment strategy is also a threshold strategy $k^*(k)$ given by*

$$\int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{k^*(k) - \theta}{\sigma}\right) \pi\left(\theta, 1 - F\left(\frac{k - \theta}{\sigma}\right)\right) d\theta = -\frac{s}{1 - t}$$

$k^*(k)$ is increasing in k.

Proof of Lemma 2. The posterior distribution of fundamental θ given signal x is

$$g(\theta | x) = \frac{1}{\sigma} f\left(\frac{x - \theta}{\sigma}\right)$$

Consider an agent who receives private signal x and knows that all other agents follow the action cutoff strategy k. The expected utility resulting from taking action 1 and reject the offer from the government is

$$U^R(k, x) = \int_{-\infty}^{\infty} g(\theta | x) \pi\left(\theta, 1 - F\left(\frac{k - \theta}{\sigma}\right)\right) d\theta$$

The expected utility resulting from taking action 1 and accept the offer from the government is

$$U^A(k, x) = (1 - t) U^R(k, x) + s$$

Therefore, the expected utility from taking action 1 is

$$U(k,x) = \max\{U^R(k,x), U^A(k,x)\}$$

Let $x = k^*(k)$ be the solution to $U^A(k,x) = 0$, i.e. the private signal such that an agent is indifferent between $\{a=0\}$ and $\{a=1, Accept\}$. Then $U^R(k^*(k,x)) = -\frac{s}{1-t} < U^A(k^*(k,x))$, and it is optimal to accept the stimulus offer at the investment threshold. Since $U(k,x)$ strictly increases in x, $k^*(k)$ is the unique investment threshold.

Note that $\forall k \in \mathbb{R}$, $\lim\limits_{\sigma\to 0} U^R(k,-\infty) \leq \pi(-\infty,1) < -\frac{s}{1-t}$ and $\lim\limits_{\sigma\to 0} U^R(k,\infty) \geq \pi(\infty,0) > 0$. In addition, $U^R(k,x)$ is continuously increasing in x. As a result, when σ is small enough, $k^*(k)$ such that $U^R(k,x) = -\frac{s}{1-t}$ must exist.

$$k^{*\prime}(k) = -\left.\frac{\frac{\partial U^R}{\partial k}}{\frac{\partial U^R}{\partial x}}\right|_{x=k^*(k)} > 0$$

■

Next prove the uniqueness of equilibrium by iterated deletion of dominated strategies. Denote the investment strategy as $a(x)$, the action choice made by an agent receiving signal x.

A strategy survives n rounds of iterated deletion of dominated strategies if and only if

$$a(x) = \begin{cases} 0, & \text{if } x < \underline{\xi}_n \\ 1, & \text{if } x > \bar{\xi}_n \end{cases}$$

where $\underline{\xi}_0 = -\infty$, $\bar{\xi}_0 = \infty$, and $\underline{\xi}_n$ and $\bar{\xi}_n$ are defined inductively by

$$\underline{\xi}_{n+1} = \min\{x : x = k^*(\underline{\xi}_n)\}$$

and

$$\bar{\xi}_{n+1} = \max\{x : x = k^*(\bar{\xi}_n)\}$$

Since $k^*(k)$ increase in k, $\underline{\xi}_n$ and $\bar{\xi}_n$ are increasing and decreasing sequences, respectively. As $n \to \infty$, $\underline{\xi}_n \to \underline{\xi}$ and $\bar{\xi}_n \to \bar{\xi}$. Therefore, $\underline{\xi} = k^*(\underline{\xi})$ and $\bar{\xi} = k^*(\bar{\xi})$.

The last step is to solve for ξ such that $\xi = k^*(\xi)$ and show uniqueness.

$$\int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{\xi-\theta}{\sigma}\right) \pi\left(\theta, 1 - F\left(\frac{\xi-\theta}{\sigma}\right)\right) d\theta = -\frac{s}{1-t}$$

By change of variable, $l = 1 - F\left(\frac{\xi-\theta}{\sigma}\right)$,

$$\int_0^1 \pi\left(\xi - \sigma F^{-1}(1-l), l\right) dl = -\frac{s}{1-t} \tag{A.38}$$

Since there is a unique solution to the equation above, $\underline{\xi} = \bar{\xi} = \xi(s,t)$. And it is the unique action cutoff in equilibrium. Given the unique investment cutoff, we can solve for the offer cutoff such that $U^A = U^R$. Denote the offer cutoff as η defined by

$$U^R(\xi, \eta) = \int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{\eta(s,t)-\theta}{\sigma}\right) \pi\left(\theta, 1 - F\left(\frac{\xi(s,t)-\theta}{\sigma}\right)\right) d\theta = \frac{s}{t}$$

The limit of left hand of equation (A.38) is

$$\lim_{\sigma\to 0} \int_0^1 \pi\left(\xi(s,t) - \sigma F^{-1}(1-l), l\right) dl = \int_0^1 \pi(\xi(s,t), l) dl$$

Therefore, $\lim\limits_{\sigma\to 0} \xi(s,t) = \xi^*$ where ξ^* is such that

$$\int_0^1 \pi(\xi^*, l) dl = -\frac{s}{1-t}$$

Since $U^R(k,x)$ decreases in k, and $U^R(\xi(s,t), \xi(s,t)) = -\frac{s}{1-t} < U^R(\xi(s,t), \eta(s,t)) = \frac{s}{t}$, $\forall \sigma > 0$ $\eta(s,t) \geq \xi(s,t)$. Therefore, $\lim\limits_{\sigma\to 0} \eta(s,t) \geq \lim\limits_{\sigma\to 0} \xi(s,t)$.

Next, prove by contradiction. Suppose $\lim\limits_{\sigma\to 0} \eta(s,t) > \lim\limits_{\sigma\to 0} \xi(s,t)$,

$$\lim_{\sigma\to 0} \int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{\eta(s,t)-\theta}{\sigma}\right) \pi\left(\theta, 1 - F\left(\frac{\xi(s,t)-\theta}{\sigma}\right)\right) d\theta = u\left(\eta, 1\right) > \frac{s}{t}$$

Contradict to

$$\int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{\eta(s,t)-\theta}{\sigma}\right) \pi\left(\theta, 1 - F\left(\frac{\xi(s,t)-\theta}{\sigma}\right)\right) d\theta = \frac{s}{t}$$

Therefore, $\lim\limits_{\sigma\to 0} \eta(s,t) = \lim\limits_{\sigma\to 0} \xi(s,t) = \xi^*$. ■

Proof of Proposition 11. According to definition 1, a partial-participation program with target ξ should satisfy the following conditions

1. $\pi(-\infty, 1) < -\frac{s}{1-t}$
2. $\pi(\xi, 1) > \frac{s}{t}$

3. $\int_0^1 \pi(\xi, l)dl = -\frac{s}{1-t}$

As long as the government offer (s, t) is such that

$$t > \left(-\frac{u(\xi, 1)}{\int_0^1 u(\xi, l)dl} + 1 \right)^{-1} \tag{A.39}$$

and

$$s = -(1-t)\int_0^1 \pi(\xi, l)dl \tag{A.40}$$

the three conditions listed above are satisfied. First, by assumption $\pi(-\infty, 1) \leq \int_0^1 \pi(\underline{\theta}, l)dl < \int_0^1 \pi(\xi, l)dl = -\frac{s}{1-t}$. Second, rearrange equation (A.39) and get $u(\xi, 1) > -\frac{1-t}{t}\int_0^1 \pi(\xi, l)dl = \frac{s}{t}$. Finally, the third condition directly follows equation (A.40). ■

Proof of Proposition 12.

$$\lim_{N\to\infty} U_N(k, x) = U^R(k, x)$$

$$U_N(\xi_N^*(s,t), \xi_N^*(s,t)) = U^R(\xi^*(s,t), \xi^*(s,t)) = -\tfrac{s}{1-t}$$
$$U_N(\xi_N^*(s,t), \eta_N^*(s,t)) = U^R(\xi^*(s,t), \eta^*(s,t)) = \tfrac{s}{t}.$$

$$\lim_{N\to\infty} \xi_N^*(s,t) = \xi^*(s,t),$$
$$\lim_{N\to\infty} \eta_N^*(s,t) = \eta^*(s,t).$$

■